

# ALMOST LOCAL METRICS ON SHAPE SPACE OF HYPERSURFACES IN $n$ -SPACE

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ABSTRACT. This paper extends parts of the results from [17] for plane curves to the case of hypersurfaces in  $\mathbb{R}^n$ . Let  $M$  be a compact connected oriented  $n - 1$  dimensional manifold without boundary like  $S^2$  or the torus  $S^1 \times S^1$ . Then shape space is either the manifold of submanifolds of  $\mathbb{R}^n$  of type  $M$ , or the orbifold of immersions from  $M$  to  $\mathbb{R}^n$  modulo the group of diffeomorphisms of  $M$ . We investigate almost local Riemannian metrics on shape space: These are induced by metrics of the following form on the space of immersions:

$$G_f(h, k) = \int_M \Phi(\text{Vol}(M), \text{Tr}(L)) \bar{g}(h, k) \text{vol}(f^* \bar{g})$$

where  $\bar{g}$  is the standard metric on  $\mathbb{R}^n$ ,  $f^* \bar{g}$  is the induced metric on  $M$ ,  $h, k \in C^\infty(M, \mathbb{R}^n)$  are tangent vectors at  $f$  to the space of embeddings or immersions, where  $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}_{>0}$  is a suitable smooth function,  $\text{Vol}(M) = \int_M \text{vol}(f^* \bar{g})$  is the total hypersurface volume of  $f(M)$ , and where the trace  $\text{Tr}(L)$  of the Weingarten mapping is the mean curvature. For these metrics we compute the geodesic equations both on the space of immersions and on shape space, the conserved momenta arising from the obvious symmetries, and the sectional curvature. For special choices of  $\Phi$  we give complete formulas for the sectional curvature. Numerical experiments illustrate the behavior of these metrics.

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## 1. INTRODUCTION

Many procedures in science, engineering, and medicine produce data in the form of shapes of point clouds in  $\mathbb{R}^n$ . If one expects such a cloud to follow roughly a submanifold of a certain type in  $\mathbb{R}^n$ , then it is of utmost importance to describe the space of all possible submanifolds of this type (we call it a shape space hereafter) and equip it with a significant metric which is able to distinguish special features of the shapes. Most of the metrics used today in data analysis and computer vision are of an ad-hoc and naive nature; one embeds shape space in some Hilbert space or Banach space and uses the distance therein. Shortest paths are then line segments, but they leave shape space quickly.

Riemannian metrics on shape space itself are a better solution. They lead to geodesics, to curvature and diffusion. Eventually one also needs statistics on shape space like means of clustered subsets of data (called Karcher means on Riemannian manifolds) and standard deviations. Here curvature will play an essential role; statistics on Riemannian manifolds seems hopelessly underdeveloped just now.

**1.1. The shape spaces used in this paper.** Thus, initially, by a shape we mean a smoothly embedded hypersurface in  $\mathbb{R}^n$  which is diffeomorphic to  $M$ . The space of these shapes will be denoted  $B_e = B_e(M, \mathbb{R}^n)$  and viewed as the quotient (see [15] for more details in a more general situation)

$$B_e(M, \mathbb{R}^n) = \text{Emb}(M, \mathbb{R}^n) / \text{Diff}(M)$$

of the open subset  $\text{Emb}(M, \mathbb{R}^n) \subset C^\infty(M, \mathbb{R}^n)$  of smooth embeddings of  $M$  in  $\mathbb{R}^n$ , modulo the group of smooth diffeomorphisms of  $M$ . It is natural to consider all possible *immersions* as well as embeddings, and thus introduce the larger space  $B_i = B_i(M, \mathbb{R}^n)$  as the quotient of the space of  $C^\infty$  immersions by the group of diffeomorphisms of  $M$  (which is, however, no longer a manifold, but an orbifold with finite isotropy groups, see [5]):

$$\begin{array}{ccccc} \text{Emb}(M, \mathbb{R}^n) & \twoheadrightarrow & \text{Emb}(M, \mathbb{R}^n) / \text{Diff}(M) & = & B_e(M, \mathbb{R}^n) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Imm}(M, \mathbb{R}^n) & \twoheadrightarrow & \text{Imm}(M, \mathbb{R}^n) / \text{Diff}(M) & = & B_i(M, \mathbb{R}^n) \end{array}$$

More generally, a shape will be an element of the Cauchy completion (i.e., the metric completion for the geodesic distance) of  $B_i(M, \mathbb{R}^n)$  with respect to a suitably chosen Riemannian metric. This will allow for corners. On the other, discretizing for numerical algorithms will hide the need to go to the Cauchy completion.

**1.2. Where this paper comes from.** In [17], Michor and Mumford have investigated a variety of Riemannian metrics on the shape space

$$B_i(S^1, \mathbb{R}^2) = \text{Imm}(S^1, \mathbb{R}^2) / \text{Diff}(S^1)$$

of immersion of the circle into the plane modulo the group of reparameterizations of the circle  $S^1$ . In [16, 3.10] they found that the simplest such metric has vanishing

geodesic distance; this is the metric induced by  $L^2$ (arc length) on  $\text{Imm}(S^1, \mathbb{R}^2)$ :

$$G_c^0(h, k) = \int_{S^1} \langle h(\theta), k(\theta) \rangle |c'(\theta)| d\theta, \\ c \in \text{Imm}(S^1, \mathbb{R}^2), \quad h, k \in C^\infty(S^1, \mathbb{R}^2) = T_c \text{Imm}(S^1, \mathbb{R}^2).$$

In [15] they found that the vanishing geodesic distance phenomenon for the  $L^2$ -metric occurs also in the more general shape space  $\text{Imm}(M, N)/\text{Diff}(M)$  where  $S^1$  is replaced by a compact manifold  $M$  and Euclidean  $\mathbb{R}^2$  is replaced by a Riemannian manifold  $N$ ; it also occurs on the full diffeomorphism group  $\text{Diff}(N)$ , but not on the subgroup  $\text{Diff}(N, \text{vol})$  of volume preserving diffeomorphisms, where the geodesic equation for the  $L^2$ -metric is the Euler equation of an incompressible fluid. In [17, section 3] a class of metrics was investigated which were called almost local metrics: they were of the form

$$G_c^\Phi(h, k) = \int_{S^1} \Phi(\ell(c), \kappa_c(\theta)) \langle h(\theta), k(\theta) \rangle |c'(\theta)| d\theta,$$

where  $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a suitable smooth function,  $\ell(c) = \int_{S^1} |c'(\theta)| d\theta$  is the length of  $c$ , and  $\kappa_c$  is the curvature of  $c$ . If  $\Phi = \Phi(\ell(c))$  then this is just a conformal change of the metric; it was proposed and investigated independently in [19] and in [23, 24, 25]. For  $\Phi = 1 + A\kappa_c^2$  the metric was investigated in great detail in [16].

In this paper we take up the investigation of almost local metrics from [17] and we generalize it to the shape space  $B_i(M, \mathbb{R}^n) = \text{Imm}(M, \mathbb{R}^n)/\text{Diff}(M)$  of hypersurfaces of type  $M$  in  $\mathbb{R}^n$ ; here  $M$  is a compact orientable connected manifold of dimension  $n - 1$ , for example the hypersphere  $S^{n-1}$ .

**1.3. Riemannian metrics.** A Riemannian metric on  $\text{Imm}(M, \mathbb{R}^n)$  is just a family of positive definite inner products  $G_f(h, k)$  where  $f \in \text{Imm}(M, \mathbb{R}^n)$  and  $h, k \in C^\infty(M, \mathbb{R}^n) = T_f \text{Imm}(M, \mathbb{R}^n)$  represent vector fields on  $\mathbb{R}^n$  along  $f$ . We require that our metrics will be invariant under the action of  $\text{Diff}(M)$ , hence the quotient map dividing by this action will be a Riemannian submersion. This means that the tangent map of the quotient map  $\text{Emb}(M, \mathbb{R}^n) \rightarrow B_i(M, \mathbb{R}^n)$  is a metric quotient mapping between all tangent spaces. Thus we will get Riemannian metrics on  $B_i$ . Here the restriction to almost local  $\text{Diff}(M)$ -invariant metrics is very beneficial: For any  $f \in \text{Emb}(M, \mathbb{R}^n)$  those vectors in  $T_f \text{Emb}(M, \mathbb{R}^n) = C^\infty(M, \mathbb{R}^n)$  which are  $G_f$ -perpendicular to the  $\text{Diff}(M)$ -orbit through  $f$  are exactly those vector fields which are pointwise normal to  $f(M)$ , i.e.,  $T_f(f \circ \text{Diff}(M))^{\perp, G_f} = \{h \in C^\infty(M, \mathbb{R}^n) : h(x) \perp T_x f(T_x M)\}$ . We shall call such vectors *horizontal*. The tangent map of the quotient map  $\text{Emb}(M, \mathbb{R}^n) \rightarrow B_i(M, \mathbb{R}^n)$  is then an isometry when restricted to the horizontal spaces, just as in the finite dimensional situation. Riemannian submersions have a very nice effect on geodesics: the geodesics on the quotient space  $B_i$  are exactly the images of the horizontal geodesics on the top space  $\text{Imm}$ . (By a horizontal geodesic we mean a geodesic whose tangent lies in the horizontal bundle.)

The simplest inner product on the tangent bundle to  $\text{Imm}(M, \mathbb{R}^n)$  is

$$G_f^0(h, k) = \int_M \bar{g}(h, k) \text{vol}(f^* \bar{g}),$$

where  $\bar{g}$  is the Euclidean inner product on  $\mathbb{R}^n$ . Since the volume form  $\text{vol}(f^*\bar{g})$  reacts equivariantly to the action of the group  $\text{Diff}(M)$ , this metric is invariant, and the map to the quotient  $B_i$  is a Riemannian submersion for this metric.

All of the metrics we will look at will be of the form:

$$G_f(h, k) = \int_M \Phi(\text{Vol}(M), \text{Tr}(L)) \bar{g}(h, k) \text{vol}(f^*\bar{g})$$

where  $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a suitable positive smooth function,  $\text{Vol}(M) = \int_M \text{vol}(f^*\bar{g})$  is the total hypersurface volume of  $f(M)$ , and where the trace  $\text{Tr}(L)$  of the Weingarten mapping is the mean curvature. As mentioned above, these metrics will be called almost local.

**1.4. The contents of this paper.** Let us now describe in some detail the contents of this paper and the metrics.

Section 2 recalls concepts from differential geometry of hypersurfaces in a form that is suitable for our needs.

In section 3 we calculate the derivatives of the metric, the volume form, the second fundamental form and some other curvature terms with respect to the immersion  $f$ , and the second derivatives for both tangent vectors horizontal. The differential calculus used is convenient calculus as in [11]. Some of the formulas can be found in [2, 4, 15], and in [21].

Section 4 describes the general Hamiltonian formalism that we shall use to compute geodesic equations and conserved quantities in a quick way. This is a shortened version of [17, section 2], updated to the more general situation here. See also [13, section 2] for a detailed exposition in similar notation. First we consider general Riemannian metrics on the space of immersions which admit Christoffel symbols. We express this as the existence of two kinds of gradients. Since the energy function is not even defined on the whole cotangent bundle of the tangent bundle we pull back to the tangent bundle the canonical symplectic structure on the cotangent bundle. Then we determine the Hamiltonian vector field mapping and, as a special case, the geodesic equation. We determine the equivariant moment mapping for several group actions on the space of immersions: The action of the reparametrization group  $\text{Diff}(M)$ , of the motion group of  $\mathbb{R}^n$ , and also of the scaling group (if the metric is scale invariant).

In section 5.1 we compute the geodesic equation on  $\text{Imm}(M, \mathbb{R}^n)$  for the general almost local metric, and its conserved momenta.

In section 6.2 we split this geodesic equation into its horizontal part (which is the geodesic equation on shape space  $B_i(M, \mathbb{R}^n)$ ) and in its vertical part which vanishes identically.

Section 7 computes the sectional curvature of  $B_i(M, \mathbb{R}^n)$  for the general almost local metric. The result is the sum of terms  $\boxed{P_1 P} \dots \boxed{P_6 P}$  and  $\int_M \frac{1}{\varphi} \dots \text{vol}(g)$  over the terms  $\boxed{Q_i Q_j}$  for  $i = 1, \dots, 5, j = 1, \dots, 5$ . Some of these terms make positive contribution to sectional curvature, some negative, and others are indefinite.

In section 8 some estimates for geodesic distance on  $B_i(M, \mathbb{R}^n)$  are given: Two area swept out bounds, and Lipschitz continuity of  $\sqrt{\text{Vol}(M)}$ . In addition we compare the almost local metrics to the Fréchet metric.

In section 9 we study the totally geodesic subspace of concentric spheres.

In section 10 we specialize the general results to special choices for the weight function  $\Phi$ . These are the following:

- The  $G^0$ -metric or  $L^2$ -metric, where  $\Phi = 1$ , geodesic distance on  $B_i$  vanishes (as shown in [15]), and which has simple non-negative sectional curvature.
- The  $G^A$ -metric where  $\Phi = 1 + A \text{Tr}(L)^2$ ; its geodesic equation was treated in [16] for the situation of plane curves, with its sectional curvature. Here we have the geodesic equation and the sectional curvature as special cases.
- Conformal metrics where  $\Phi = \Phi(\text{Vol})$ . For curves these were investigated in [19, 23, 24, 25]. The full formula for sectional curvature is given only in the case that  $\Phi(\text{Vol}) = \text{Vol}$ .
- The scale invariant version where  $\Phi = \text{Vol}^{\frac{1+n}{1-n}} + A \frac{\text{Tr}(L)^2}{\text{Vol}}$ .

Section 11 contains numerical experiments for geodesics. We only do boundary value problems and no initial value problems (it is not clear that the equations are well posed). We use Mathematica to set up the triangulation of the surfaces, feed this into AMPL (a modelling software developed for optimization) and use the solver IPOPT. The numerical results are tested on the totally geodesic subspace of concentric spheres where we also have the analytic solutions. Then we study the translations of spheres for various metrics and discuss the appearing phenomena. Finally we deform some surfaces.

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Big parts of this paper can be found in the doctoral dissertation of Martin Bauer, [3].

## 2. DIFFERENTIAL GEOMETRY OF SURFACES AND NOTATION

In this section we will present and develop the differential geometric tools that are needed to deal with immersed surfaces. The most important point is a rigorous treatment of the covariant derivative and related concepts.

In [2, section 2] one can find some parts of this section in a more general setting. We use the notation of [14]. Some of the definitions can also be found in [9]. A similar exposition in the same notation is [1].

**2.1. Basic assumptions and convention.** We always assume that  $M$  is a connected and compact manifold of dimension  $n - 1$  without boundary. We will work with *immersions* of  $M$  into  $\mathbb{R}^n$ , i.e. smooth functions  $M \rightarrow \mathbb{R}^n$  with injective tangent mapping at every point. We denote the set of all such immersions by  $\text{Imm}(M, \mathbb{R}^n)$ . Immersions or paths of immersions are usually denoted by  $f$ . Vector

fields on  $\text{Imm}(M, \mathbb{R}^n)$  or vector fields along  $f$  will be called  $h, k, m$ , for example. Subscripts like  $f_t$  denote differentiation with respect to the indicated variable, so  $f_t = \partial_t f = \partial f / \partial t$ , but subscripts are also used to indicate the foot point of a tensor field.

**2.2. Tensor bundles and tensor fields.** We will deal with the *tensor bundles*

$$\begin{array}{ccc} T_s^r M & & T_s^r M \otimes f^* T\mathbb{R}^n \\ \downarrow & & \downarrow \\ M & & M \end{array}$$

Here  $T_s^r M$  denotes the bundle of  $\binom{r}{s}$ -tensors on  $M$ , i.e.

$$T_s^r M = \bigotimes^r TM \otimes \bigotimes^s T^*M,$$

and  $f^* T\mathbb{R}^n$  is the pullback of the bundle  $T\mathbb{R}^n$  via  $f$ , see [14, section 17.5]. A *tensor field* is a section of a tensor bundle. Generally, when  $E$  is a bundle, the space of its sections will be denoted by  $\Gamma(E)$ .

To clarify the notation that will be used later, some examples of tensor bundles and tensor fields are given now.

- $S^k T^* M = L_{\text{sym}}^k(TM; \mathbb{R})$  is the bundle of *symmetric*  $\binom{0}{k}$ -tensors,
- $\Lambda^k T^* M = L_{\text{alt}}^k(TM; \mathbb{R})$  is the bundle of *alternating*  $\binom{0}{k}$ -tensors,
- $\Omega^r(M) = \Gamma(\Lambda^r T^* M)$  is the space of *differential forms*,
- $\mathfrak{X}(M) = \Gamma(TM)$  is the space of *vector fields*, and
- $\Gamma(f^* T\mathbb{R}^n) \cong \{h \in C^\infty(M, T\mathbb{R}^n) : \pi_N \circ h = f\}$  is the space of *vector fields along  $f$* .

**2.3. Metric on tensor spaces.** Let  $\bar{g} \in \Gamma(S_{>0}^2 T^* \mathbb{R}^n)$  denote the Euclidean metric on  $\mathbb{R}^n$ . The *metric induced on  $M$  by  $f \in \text{Imm}(M, \mathbb{R}^n)$*  is the pullback metric

$$g = f^* \bar{g} \in \Gamma(S_{>0}^2 T^* M), \quad g(X, Y) = (f^* \bar{g})(X, Y) = \bar{g}(Tf.X, Tf.Y),$$

where  $X, Y$  are vector fields on  $M$ . The dependence of  $g$  on the immersion  $f$  should be kept in mind. Let

$$\flat = \check{g} : TM \rightarrow T^*M \quad \text{and} \quad \sharp = \check{g}^{-1} : T^*M \rightarrow TM.$$

$g$  can be extended to the cotangent bundle  $T^*M = T_1^0 M$  by setting

$$g^{-1}(\alpha, \beta) = g_1^0(\alpha, \beta) = \alpha(\beta^\sharp)$$

for  $\alpha, \beta \in T^*M$ . The product metric

$$g_s^r = \bigotimes^r g \otimes \bigotimes^s g^{-1}$$

extends  $g$  to all tensor spaces  $T_s^r M$ , and  $g_s^r \otimes \bar{g}$  yields a metric on  $T_s^r M \otimes f^* T\mathbb{R}^n$ .

**2.4. Traces.** The *trace* contracts pairs of vectors and co-vectors in a tensor product:

$$\text{Tr} : T^*M \otimes TM = L(TM, TM) \rightarrow M \times \mathbb{R}$$

A special case of this is the operator  $i_X$  inserting a vector  $X$  into a co-vector or into a covariant factor of a tensor product. The inverse of the metric  $g$  can be used to define a trace

$$\text{Tr}^g : T^*M \otimes T^*M \rightarrow M \times \mathbb{R}$$

contracting pairs of co-vectors. Note that  $\text{Tr}^g$  depends on the metric whereas  $\text{Tr}$  does not. The following lemma will be useful in many calculations (see [2, section 2]):

**Lemma.**

$$g_2^0(B, C) = \text{Tr}(g^{-1}Bg^{-1}C) \quad \text{for } B, C \in T_2^0M \text{ if } B \text{ or } C \text{ is symmetric.}$$

(In the expression under the trace,  $B$  and  $C$  are seen maps  $TM \rightarrow T^*M$ .)

**2.5. Volume density.** Let  $\text{Vol}(M)$  be the *density bundle* over  $M$ , see [14, section 10.2]. The *volume density* on  $M$  induced by  $f \in \text{Imm}(M, \mathbb{R}^n)$  is

$$\text{vol}(g) = \text{vol}(f^*\bar{g}) \in \Gamma(\text{Vol}(M)).$$

The *volume* of the immersion is given by

$$\text{Vol}(f) = \int_M \text{vol}(f^*\bar{g}) = \int_M \text{vol}(g).$$

The integral is well-defined since  $M$  is compact. If  $M$  is oriented we may identify the volume density with a differential form.

**2.6. Covariant derivative.** We will use covariant derivatives on vector bundles as explained in [14, sections 19.12, 22.9]. Let  $\nabla^g, \nabla^{\bar{g}}$  be the *Levi-Civita covariant derivatives* on  $(M, g)$  and  $(\mathbb{R}^n, \bar{g})$ , respectively. For any manifold  $Q$  and vector field  $X$  on  $Q$ , one has

$$\begin{aligned} \nabla_X^g : C^\infty(Q, TM) &\rightarrow C^\infty(Q, TM), & h &\mapsto \nabla_X^g h \\ \nabla_X^{\bar{g}} : C^\infty(Q, T\mathbb{R}^n) &\rightarrow C^\infty(Q, T\mathbb{R}^n), & h &\mapsto \nabla_X^{\bar{g}} h. \end{aligned}$$

Usually we will simply write  $\nabla$  for all covariant derivatives. It should be kept in mind that  $\nabla^g$  depends on the metric  $g = f^*\bar{g}$  and therefore also on the immersion  $f$ . The following properties hold [14, section 22.9]:

- (1)  $\nabla_X$  respects base points, i.e.  $\pi \circ \nabla_X h = \pi \circ h$ , where  $\pi$  is the projection of the tangent space onto the base manifold.
- (2)  $\nabla_X h$  is  $C^\infty$ -linear in  $X$ . So for a tangent vector  $X_x \in T_x Q$ ,  $\nabla_{X_x} h$  makes sense and equals  $(\nabla_X h)(x)$ .
- (3)  $\nabla_X h$  is  $\mathbb{R}$ -linear in  $h$ .
- (4)  $\nabla_X(a \cdot h) = da(X) \cdot h + a \cdot \nabla_X h$  for  $a \in C^\infty(Q)$ , the derivation property of  $\nabla_X$ .
- (5) For any manifold  $\tilde{Q}$  and smooth mapping  $q : \tilde{Q} \rightarrow Q$  and  $Y_y \in T_y \tilde{Q}$  one has  $\nabla_{Tq \cdot Y_y} h = \nabla_{Y_y}(h \circ q)$ . If  $Y \in \mathfrak{X}(\tilde{Q})$  and  $X \in \mathfrak{X}(Q)$  are  $q$ -related, then  $\nabla_Y(h \circ q) = (\nabla_X h) \circ q$ .

The two covariant derivatives  $\nabla_X^g$  and  $\nabla_X^{\bar{g}}$  can be combined to yield a covariant derivative  $\nabla_X$  acting on  $C^\infty(Q, T_s^r M \otimes T\mathbb{R}^n)$  by additionally requiring the following properties [14, section 22.12]:

- (6)  $\nabla_X$  respects the spaces  $C^\infty(Q, T_s^r M \otimes TN)$ .
- (7)  $\nabla_X(h \otimes k) = (\nabla_X h) \otimes k + h \otimes (\nabla_X k)$ , a derivation with respect to the tensor product.
- (8)  $\nabla_X$  commutes with any kind of contraction (see [14, section 8.18]). A special case of this is

$$\nabla_X(\alpha(Y)) = (\nabla_X \alpha)(Y) + \alpha(\nabla_X Y) \quad \text{for } \alpha \otimes Y : \mathbb{R}^n \rightarrow T_1^1 M.$$

Property (1) is important because it implies that  $\nabla_X$  respects spaces of sections of bundles. For example, for  $Q = M$  and  $f \in C^\infty(M, \mathbb{R}^n)$ , one gets

$$\nabla_X : \Gamma(T_s^r M \otimes f^* T\mathbb{R}^n) \rightarrow \Gamma(T_s^r M \otimes f^* T\mathbb{R}^n).$$

**2.7. Swapping covariant derivatives.** We will make repeated use of some formulas allowing to swap covariant derivatives. Let  $f$  be an immersion,  $h$  a vector field along  $f$  and  $X, Y$  vector fields on  $M$ . Since  $\nabla$  is torsion-free, one has [14, section 22.10]

$$(1) \quad \nabla_X T f.Y - \nabla_Y T f.X - T f.[X, Y] = \text{Tor}(T f.X, T f.Y) = 0.$$

Furthermore one has [14, section 24.5]

$$(2) \quad \nabla_X \nabla_Y h - \nabla_Y \nabla_X h - \nabla_{[X, Y]} h = R^{\bar{g}} \circ (T f.X, T f.Y) h = 0.$$

These formulas also hold when  $f : \mathbb{R} \times M \rightarrow \mathbb{R}^n$  is a path of immersions,  $h : \mathbb{R} \times M \rightarrow T\mathbb{R}^n$  is a vector field along  $f$  and the vector fields are vector fields on  $\mathbb{R} \times M$ . A case of special importance is when one of the vector fields is  $(\partial_t, 0_M)$  and the other  $(0_{\mathbb{R}}, Y)$ , where  $Y$  is a vector field on  $M$ . Since the Lie bracket of these vector fields vanishes, (1) and (2) yield

$$(3) \quad \nabla_{(\partial_t, 0_M)} T f.(0_{\mathbb{R}}, Y) - \nabla_{(0_{\mathbb{R}}, Y)} T f.(\partial_t, 0_M) = 0$$

and

$$(4) \quad \nabla_{(\partial_t, 0_M)} \nabla_{(0_{\mathbb{R}}, Y)} h - \nabla_{(0_{\mathbb{R}}, Y)} \nabla_{(\partial_t, 0_M)} h = 0.$$

**2.8. Higher covariant derivatives and the Laplace operator.** When the covariant derivative is seen as a mapping

$$\nabla : \Gamma(T_s^r M) \rightarrow \Gamma(T_{s+1}^r M) \quad \text{or} \quad \nabla : \Gamma(T_s^r M \otimes f^* T\mathbb{R}^n) \rightarrow \Gamma(T_{s+1}^r M \otimes f^* T\mathbb{R}^n),$$

then the *second covariant derivative* is simply  $\nabla \nabla = \nabla^2$ . Since the covariant derivative commutes with contractions,  $\nabla^2$  can be expressed as

$$\nabla_{X,Y}^2 := \iota_Y \iota_X \nabla^2 = \iota_Y \nabla_X \nabla = \nabla_X \nabla_Y - \nabla_{\nabla_X Y} \quad \text{for } X, Y \in \mathfrak{X}(M).$$

Higher covariant derivatives are defined as  $\nabla^k$ ,  $k \geq 0$ . We can use the second covariant derivative to define the *Laplace-Bochner operator*. It can act on all tensor fields  $B$  and is defined as

$$\Delta B = -\text{Tr}^g(\nabla^2 B).$$



**2.9. Normal bundle.** The *normal bundle*  $\text{Nor}(f)$  of an immersion  $f$  is a sub-bundle of  $f^*T\mathbb{R}^n$  whose fibers consist of all vectors that are orthogonal to the image of  $f$ :

$$\text{Nor}(f)_x = \{Y \in T_{f(x)}N : \forall X \in T_xM : \bar{g}(Y, Tf.X) = 0\}.$$

Any vector field  $h$  along  $f$  can be decomposed uniquely into parts *tangential* and *normal* to  $f$  as

$$h = Tf.h^\top + h^\perp,$$

where  $h^\top$  is a vector field on  $M$  and  $h^\perp$  is a section of the normal bundle  $\text{Nor}(f)$ . When  $f$  is orientable, then the unit normal field  $\nu$  of  $f$  can be defined. It is a section of the normal bundle in one of the above forms with constant  $\bar{g}$ -length one which is chosen such that

$$(\nu(x), Tf.X_1, Tf.X_2, \dots, Tf.X_{n-1})$$

is a positive oriented basis in  $T_{f(x)}\mathbb{R}^n$  if  $X_1, \dots, X_{n-1}$  is a positive oriented basis in  $T_xM$ . In this notation the decomposition of a vector field  $h$  along  $f$  reads as

$$h = Tf.h^\top + a.\nu.$$

The two parts are defined by the relations

$$a = \bar{g}(h, \nu) \in C^\infty(M)$$

$$h^\top \in \mathfrak{X}(M), \text{ such that } g(h^\top, X) = \bar{g}(h, Tf(t, \cdot).X) \text{ for all } X \in \mathfrak{X}(M).$$

**2.10. Second fundamental form and Weingarten mapping.** Let  $X$  and  $Y$  be vector fields on  $M$ . Then the covariant derivative  $\nabla_X Tf.Y$  splits into tangential and a normal parts as

$$\nabla_X Tf.Y = Tf.(\nabla_X Tf.Y)^\top + (\nabla_X Tf.Y)^\perp = Tf.\nabla_X Y + S(X, Y).$$

$S$  is the *second fundamental form* of  $f$ . It is a symmetric bilinear form with values in the normal bundle of  $f$ . When  $Tf$  is seen as a section of  $T^*M \otimes f^*T\mathbb{R}^n$  one has  $S = \nabla Tf$  since

$$S(X, Y) = \nabla_X Tf.Y - Tf.\nabla_X Y = (\nabla Tf)(X, Y).$$

Taking the trace of  $S$  yields the *vector valued mean curvature*

$$\text{Tr}^g(S) \in \Gamma(\text{Nor}(f)).$$

One can define the *scalar second fundamental form*  $s$  as

$$s(X, Y) = \bar{g}(S(X, Y), \nu).$$

Moreover, there is the *Weingarten mapping* or *shape operator*  $L = g^{-1}s$ . It is a  $g$ -symmetric bundle mapping defined by

$$s(X, Y) = g(LX, Y).$$

The eigenvalues of  $L$  are called *principal curvatures* and the eigenvectors *principal curvature directions*.  $\text{Tr}(L) = \text{Tr}^g(s)$  is the *scalar mean curvature* and for surfaces in  $\mathbb{R}^3$  the *Gauß-curvature* is given by  $\det(L)$ . The covariant derivative  $\nabla_X \nu$  of the normal vector is related to  $L$  by the *Weingarten equation*

$$\nabla_X \nu = -Tf.LX.$$

**2.11. Directional derivatives of functions.** We will use the following ways to denote directional derivatives of functions, in particular in infinite dimensions. Given a function  $F(x, y)$  for instance, we will write:

$$D_{(x,h)}F \text{ as shorthand for } \partial_t|_0 F(x + th, y).$$

Here  $(x, h)$  in the subscript denotes the tangent vector with foot point  $x$  and direction  $h$ . If  $F$  takes values in some linear space, we will identify this linear space and its tangent space.

### 3. FORMULAS FOR FIRST VARIATIONS

Recall that many operators like

$$g = f^* \bar{g}, \quad S = S^f, \quad \text{vol}(g), \quad \nabla = \nabla^g, \quad \Delta = \Delta^g, \quad \dots$$

implicitly depend on the immersion  $f$ . We want to calculate their derivative with respect to  $f$ , which we call *the first variation*. We will use this formulas to calculate the metric gradients that are needed for the geodesic equation.

Some of the formulas can also be found in [2, 4, 15, 21, 8].

**3.1. Paths of immersions.** All of the concepts introduced in section 2 can be recast for a path of immersions instead of a fixed immersion. This allows to study variations immersions. So let  $f : \mathbb{R} \rightarrow \text{Imm}(M, N)$  be a path of immersions. By convenient calculus [11],  $f$  can equivalently be seen as  $f : \mathbb{R} \times M \rightarrow N$  such that  $f(t, \cdot)$  is an immersion for each  $t$ . We can replace bundles over  $M$  by bundles over  $\mathbb{R} \times M$ :

$$\begin{array}{ccc} \text{pr}_2^* T_s^r M & \text{pr}_2^* T_s^r M \otimes f^* TN & \text{Nor}(f) \\ \downarrow & \downarrow & \downarrow \\ \mathbb{R} \times M & \mathbb{R} \times M & \mathbb{R} \times M \end{array}$$

Here  $\text{pr}_2$  denotes the projection  $\text{pr}_2 : \mathbb{R} \times M \rightarrow M$ . The covariant derivative  $\nabla_Z h$  is now defined for vector fields  $Z$  on  $\mathbb{R} \times M$  and sections  $h$  of the above bundles. The vector fields  $(\partial_t, 0_M)$  and  $(0_{\mathbb{R}}, X)$ , where  $X$  is a vector field on  $M$ , are of special importance. Let

$$\text{ins}_t : M \rightarrow \mathbb{R} \times M, \quad x \mapsto (t, x).$$

Then by [14, 22.9.6] one has for vector fields  $X, Y$  on  $M$

$$\begin{aligned} \nabla_X T f(t, \cdot).Y &= \nabla_X T(f \circ \text{ins}_t) \circ Y = \nabla_X T f \circ T \text{ins}_t \circ Y \\ &= \nabla_X T f \circ (0_{\mathbb{R}}, Y) \circ \text{ins}_t = \nabla_{T \text{ins}_t \circ X} T f \circ (0_{\mathbb{R}}, Y) \\ &= (\nabla_{(0_{\mathbb{R}}, X)} T f \circ (0_{\mathbb{R}}, Y)) \circ \text{ins}_t. \end{aligned}$$

This shows that one can recover the static situation at  $t$  by using vector fields on  $\mathbb{R} \times M$  with vanishing  $\mathbb{R}$ -component and evaluating at  $t$ .

**3.2. Setting for first variations.** In all of this chapter, let  $f$  be an immersion and  $f_t \in T_f \text{Imm}$  a tangent vector to  $f$ . The reason for calling the tangent vector  $f_t$  is that in calculations it will often be the derivative of a curve of immersions through  $f$ . Using the same symbol  $f$  for the fixed immersion and for the path of immersions through it, one has in fact that

$$D_{(f, f_t)} F = \partial_t F(f(t)).$$

For the sake of brevity we will write  $\partial_t$  instead of  $(\partial_t, 0_M)$  and  $X$  instead of  $(0_{\mathbb{R}}, X)$ , where  $X$  is a vector field on  $M$ .

Let the smooth mapping  $F : \text{Imm}(M, N) \rightarrow \Gamma(T_s^r M)$  take values in some space of tensor fields over  $M$ , or more generally in any natural bundle over  $M$ , see [10].

**3.3. Lemma** (Tangential variation of equivariant tensor fields). *If  $F$  is equivariant with respect to pullbacks by diffeomorphisms of  $M$ , i.e.*

$$F(f) = (\varphi^* F)(f) = \varphi^* \left( F((\varphi^{-1})^* f) \right)$$

for all  $\varphi \in \text{Diff}(M)$  and  $f \in \text{Imm}(M, N)$ , then the tangential variation of  $F$  is its Lie-derivative:

$$\begin{aligned} D_{(f, Tf \cdot f_t^\top)} F &= \partial_t|_0 F \left( f \circ Fl_t^{f_t^\top} \right) = \partial_t|_0 F \left( (Fl_t^{f_t^\top})^* f \right) \\ &= \partial_t|_0 \left( Fl_t^{f_t^\top} \right)^* (F(f)) = \mathcal{L}_{f_t^\top} (F(f)). \end{aligned}$$

This allows us to calculate the tangential variation of the pullback metric and the volume density, because these tensor fields are natural with respect to pullbacks by diffeomorphisms.

**3.4. Lemma** (Variation of the metric). *The differential of the pullback metric*

$$\begin{cases} \text{Imm} & \rightarrow \Gamma(S_{>0}^2 T^* M), \\ f & \mapsto g = f^* \bar{g} \end{cases}$$

is given by

$$D_{(f, f_t)} g = -2\bar{g}(f_t, \nu) \cdot s + \mathcal{L}_{f_t^\top}(g).$$

*Proof.* Let  $f : \mathbb{R} \times M \rightarrow N$  be a path of immersions. Swapping covariant derivatives as in section 2.7 formula (3) one gets

$$\begin{aligned} \partial_t(g(X, Y)) &= \partial_t(\bar{g}(Tf \cdot X, Tf \cdot Y)) = \bar{g}(\nabla_{\partial_t} Tf \cdot X, Tf \cdot Y) + \bar{g}(Tf \cdot X, \nabla_{\partial_t} Tf \cdot Y) \\ &= \bar{g}(\nabla_X f_t, Tf \cdot Y) + \bar{g}(Tf \cdot X, \nabla_Y f_t) = (2 \text{Sym } \bar{g}(\nabla f_t, Tf))(X, Y). \end{aligned}$$

Splitting  $f_t$  into its normal and tangential part yields

$$\begin{aligned} 2 \text{Sym } \bar{g}(\nabla f_t, Tf) &= 2 \text{Sym } \bar{g}(\nabla f_t^\perp + \nabla Tf \cdot f_t^\top, Tf) \\ &= -2 \text{Sym } \bar{g}(f_t^\perp, \nabla Tf) + 2 \text{Sym } g(\nabla f_t^\top, \cdot) \\ &= -2\bar{g}(f_t^\perp, S) + 2 \text{Sym } \nabla(f_t^\top)^\flat. \end{aligned}$$

Finally the relation

$$D_{(f, Tf \cdot f_t^\top)} g = 2 \text{Sym } \nabla(f_t^\top)^\flat = \mathcal{L}_{f_t^\top} g$$

follows from the equivariance of  $g$  (see 3.3).  $\square$

**3.5. Lemma** (Variation of the inverse of the metric). *The differential of the inverse of the pullback metric*

$$\begin{cases} \text{Imm} & \rightarrow \Gamma(L(T^*M, TM)), \\ f & \mapsto g^{-1} = (f^*\bar{g})^{-1} \end{cases}$$

is given by

$$D_{(f, f_t)} g^{-1} = -2\bar{g}(f_t, \nu).L.g^{-1} + \mathcal{L}_{f_t^\top}(g^{-1}).$$

*Proof.*

$$\begin{aligned} \partial_t g^{-1} &= -g^{-1}(\partial_t g)g^{-1} = -g^{-1}(-2\bar{g}(f_t^\perp, S) + \mathcal{L}_{f_t^\top} g)g^{-1} \\ &= 2g^{-1}\bar{g}(f_t^\perp, S)g^{-1} - g^{-1}(\mathcal{L}_{f_t^\top} g)g^{-1} = 2\bar{g}(f_t^\perp, g^{-1}Sg^{-1}) + \mathcal{L}_{f_t^\top}(g^{-1}) \quad \square \end{aligned}$$

**3.6. Lemma** (Variation of the volume density). *The differential of the volume density*

$$\begin{cases} \text{Imm} & \rightarrow \text{Vol}(M), \\ f & \mapsto \text{vol}(g) = \text{vol}(f^*\bar{g}) \end{cases}$$

is given by

$$D_{(f, f_t)} \text{vol}(g) = \left( \text{div}^g(f_t^\top) - \bar{g}(f_t^\perp, \nu). \text{Tr}(L) \right) \text{vol}(g).$$

*Proof.* Let  $g(t) \in \Gamma(S_{>0}^2 T^*M)$  be any curve of Riemannian metrics. Then

$$\partial_t \text{vol}(g) = \frac{1}{2} \text{Tr}(g^{-1}. \partial_t g) \text{vol}(g).$$

This follows from the formula for  $\text{vol}(g)$  in a local oriented chart  $(u^1, \dots, u^n)$  on  $M$ :

$$\begin{aligned} \partial_t \text{vol}(g) &= \partial_t \sqrt{\det((g_{ij})_{ij})} du^1 \wedge \dots \wedge du^{n-1} \\ &= \frac{1}{2\sqrt{\det((g_{ij})_{ij})}} \text{Tr}(\text{adj}(g)\partial_t g) du^1 \wedge \dots \wedge du^{n-1} \\ &= \frac{1}{2\sqrt{\det((g_{ij})_{ij})}} \text{Tr}(\det((g_{ij})_{ij})g^{-1}\partial_t g) du^1 \wedge \dots \wedge du^{n-1} \\ &= \frac{1}{2} \text{Tr}(g^{-1}. \partial_t g) \text{vol}(g) \end{aligned}$$

Now we can set  $g = f^*\bar{g}$  and plug in the formula for  $\partial_t g = \partial_t(f^*\bar{g})$ . This yields

$$\begin{aligned} \partial_t \text{vol}(g) &= \frac{1}{2} \text{Tr}(g^{-1}(-2\bar{g}(f_t, \nu).s + \mathcal{L}_{h^\top} g)). \text{vol}(g) \\ &= -\bar{g}(f_t, \nu) \text{Tr}(g^{-1}.s). \text{vol}(g) + \frac{1}{2} \text{Tr}(g^{-1}\mathcal{L}_{h^\top} g). \text{vol}(g) \end{aligned}$$

The same calculation as above with  $\partial_t$  replaced by  $\mathcal{L}_{h^\top}$  shows that

$$\mathcal{L}_{h^\top} \text{vol}(g) = \frac{1}{2} \text{Tr}(g^{-1}\mathcal{L}_{h^\top} g). \text{vol}(g).$$

Therefore

$$\begin{aligned} \partial_t \text{vol}(g) &= -\bar{g}(f_t, \nu) \text{Tr}(L). \text{vol}(g) + \mathcal{L}_{h^\top}(\text{vol}(g)) \\ &= -\bar{g}(f_t, \nu) \text{Tr}(L). \text{vol}(g) + \text{div}^g(h^\top) \text{vol}(g). \quad \square \end{aligned}$$

**3.7. Lemma** (Variation of the volume). *The differential of the total Volume*

$$\begin{cases} \text{Imm} & \rightarrow \mathbb{R}, \\ f & \mapsto \text{Vol}(f) = \int_M \text{vol}(f^* \bar{g}) \end{cases}$$

is given by

$$D_{(f, f_t)} \text{vol}(g) = - \int_M \bar{g}(f_t^\perp, \nu) \cdot \text{Tr}(L) \text{vol}(g).$$

**3.8. Lemma** (Variation of the second fundamental form). *The differential of the second fundamental form*

$$\begin{cases} \text{Imm} & \rightarrow \Gamma(S^2 T^* M), \\ f & \mapsto s^f \end{cases}$$

is given by

$$D_{(f, f_t)} s = \bar{g}(\nabla^2 f_t, \nu) = \nabla^2 \bar{g}(f_t, \nu) - \bar{g}(f_t, \nu) \cdot g \circ (L \otimes L) + \mathcal{L}_{f_t^\top} s$$

*Proof.* By definition  $s(X, Y) = \bar{g}(S(X, Y), \nu) = \bar{g}(\nabla_X(Tf \cdot Y) - Tf \cdot \nabla_X Y, \nu)$ . Interchanging covariant derivatives as in 2.7.(3) and 2.7.(4) yields:

$$\begin{aligned} \partial_t s(X, Y) &= \bar{g}(\partial_t S(X, Y), \nu) + \bar{g}(S(X, Y), \partial_t \nu) \\ &= \bar{g}(\nabla_X \nabla_Y Tf \cdot \partial_t - \nabla_{\nabla_X Y} Tf \cdot \partial_t, \nu) + 0 = \bar{g}(\nabla_{X, Y}^2 f_t, \nu) \end{aligned}$$

where the term  $\bar{g}(S(X, Y), \partial_t \nu)$  vanishes since  $\partial_t \nu$  is tangential (see 3.11). For the normal part this yields: To get the second formula we calculate:

$$\begin{aligned} D_{(f, \bar{g}(f_t, \nu) \cdot \nu)} s(X, Y) &= \bar{g}(\nabla_{X, Y}^2 (\bar{g}(f_t, \nu) \cdot \nu), \nu) \\ &= \nabla_{X, Y}^2 \bar{g}(f_t, \nu) + 0 + \bar{g}(f_t, \nu) \cdot \bar{g}(\nabla_{X, Y}^2 \nu, \nu) \\ &= \nabla_{X, Y}^2 \bar{g}(f_t, \nu) - \bar{g}(f_t, \nu) \cdot \bar{g}(\nabla_X \nu, \nabla_Y \nu) + 0 \\ &= \nabla_{X, Y}^2 \bar{g}(f_t, \nu) - \bar{g}(f_t, \nu) \cdot g(LX, LY) \end{aligned}$$

By 3.3, the formula for the tangential variation follows from the equivariance of the second fundamental form:

$$\begin{aligned} s^{f \circ \phi}(X, Y) &= \bar{g}(\bar{\nabla}_X T(f \circ \phi) \circ Y, \nu^{f \circ \phi}) = \bar{g}(\bar{\nabla}_X (Tf \circ (\phi_* Y) \circ \phi), \nu^f \circ \phi) \\ &= \bar{g}(\bar{\nabla}_{T\phi \circ X} Tf \circ (\phi_* Y), \nu^f \circ \phi) \circ \phi \\ &= \bar{g}(\bar{\nabla}_{\phi_* X} Tf \circ (\phi_* Y), \nu^f) \circ \phi = s^f(\phi_* X, \phi_* Y) \circ \phi = (\phi^* s^f)(X, Y) \quad \square \end{aligned}$$

**3.9. Lemma** (Variation of the Weingarten map). *The differential of the Weingarten map*

$$\begin{cases} \text{Imm} & \rightarrow \Gamma(\text{End}(TM)), \\ f & \mapsto L^f \end{cases}$$

is given by

$$D_{(f, f_t)} L = g^{-1} \cdot \nabla^2(\bar{g}(f_t, \nu)) + \bar{g}(f_t, \nu) L^2 + \mathcal{L}_{f_t^\top}(L).$$

*Proof.* From  $L = g^{-1} \cdot s$  follows

$$\begin{aligned} \partial_t L &= g^{-1} \cdot \partial_t s + \partial_t(g^{-1}) \cdot s \\ &= g^{-1} \cdot \left( \nabla^2(\bar{g}(f_t, \nu)) - \bar{g}(f_t, \nu) g \cdot L^2 + \mathcal{L}_{f_t^\top}(s) \right) + \left( 2\bar{g}(f_t, \nu) L g^{-1} + \mathcal{L}_{f_t^\top}(g^{-1}) \right) \cdot s \\ &= g^{-1} \cdot \nabla^2(\bar{g}(f_t, \nu)) + \bar{g}(f_t, \nu) \cdot L^2 + \mathcal{L}_{f_t^\top}(L). \quad \square \end{aligned}$$

**3.10. Lemma** (Variation of the mean curvature). *The differential of the mean curvature*

$$\begin{cases} \text{Imm} & \rightarrow C^\infty(M), \\ f & \mapsto \text{Tr}(L^f) \end{cases}$$

is given by

$$D_{(f,f_t)} \text{Tr}(L) = -\Delta(\bar{g}(f_t, \nu)) + \bar{g}(f_t, \nu) \cdot \text{Tr}(L^2) + d(\text{Tr}(L))(f_t^\top).$$

*Proof.* This statement follows from the linearity of the trace operator and from the previous equation for  $D_{(f,f_t)}L$ .  $\square$

**3.11. Lemma** (Variation of the normal vector field). *The normal vector field is a smooth map  $\nu : \mathbb{R} \times M \rightarrow \mathbb{R}^n$ . Therefore, as explained in section 2.6, we can take its covariant derivative along vector fields on  $\mathbb{R} \times M$ . Identifying  $\partial_t$  with the vector field  $(\partial_t, 0_M)$  on  $\mathbb{R} \times M$ , we get*

$$\nabla_{\partial_t} \nu = -Tf \cdot (Lf_t^\top + \text{grad}^g(\bar{g}(f_t, \nu))).$$

*Proof.*  $\nabla_{\partial_t} \nu$  is tangential because  $\bar{g}(\nabla_{\partial_t} \nu, \nu) = \frac{1}{2} \partial_t \bar{g}(\nu, \nu) = 0$ . Therefore one can write  $\nabla_{\partial_t} \nu = Tf \cdot (\nabla_{\partial_t} \nu)^\top$ . Then for all  $X \in \mathfrak{X}(M)$  we have

$$g((\nabla_{\partial_t} \nu)^\top, X) = \bar{g}(\nabla_{\partial_t} \nu, Tf \cdot X) = 0 - \bar{g}(\nu, \nabla_{\partial_t} Tf \cdot X) = -\bar{g}(\nu, \nabla_X Tf \cdot \partial_t),$$

where in the last step we swapped  $X$  and  $\partial_t$  as in section 2.7 formula (3). Splitting into normal and tangential parts yield:

$$\begin{aligned} g((\nabla_{\partial_t} \nu)^\top, X) &= -\bar{g}(\nu, \nabla_X f_t) = -\bar{g}(\nu, \nabla_X (Tf \cdot f_t^\top + \bar{g}(f_t, \nu) \cdot \nu)) \\ &= -\bar{g}(\nu, \nabla_X (Tf \cdot f_t^\top + \bar{g}(f_t, \nu) \cdot \nu)) \\ &= -s(X, f_t^\top) - \nabla_X (\bar{g}(f_t, \nu)) - 0 \\ &= -g(Lf_t^\top + \text{grad}^g \bar{g}(f_t, \nu), X) \end{aligned} \quad \square$$

In this section, let  $\nabla = \nabla^g = \nabla^{f^* \bar{g}}$  be the Levi-Civita covariant derivative acting on vector fields on  $M$ . Since any two covariant derivatives on  $M$  differ by a tensor field, the first variation of  $\nabla^{f^* \bar{g}}$  is tensorial. It is given by the tensor field  $D_{(f,f_t)} \nabla^{f^* \bar{g}} \in \Gamma(T_2^1 M)$ .

**3.12. Lemma** (Variation of the covariant derivative). *The tensor field  $D_{(f,f_t)} \nabla^{f^* \bar{g}}$  is determined by the following relation holding for vector fields  $X, Y, Z$  on  $M$ :*

$$g((D_{(f,f_t)} \nabla)(X, Y), Z) = \frac{1}{2} (\nabla D_{(f,f_t)} g)(X \otimes Y \otimes Z + Y \otimes X \otimes Z - Z \otimes X \otimes Y)$$

*Proof.* The defining formula for the covariant derivative is

$$\begin{aligned} g(\nabla_X Y, Z) &= \frac{1}{2} [Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y])]. \end{aligned}$$

Taking the derivative  $D_{(f,f_t)}$  yields

$$(D_{(f,f_t)} g)(\nabla_X Y, Z) + g((D_{(f,f_t)} \nabla)(X, Y), Z)$$

$$= \frac{1}{2} \left[ X((D_{(f,f_t)}g)(Y, Z)) + Y((D_{(f,f_t)}g)(Z, X)) - Z((D_{(f,f_t)}g)(X, Y)) \right. \\ \left. - (D_{(f,f_t)}g)(X, [Y, Z]) + (D_{(f,f_t)}g)(Y, [Z, X]) + (D_{(f,f_t)}g)(Z, [X, Y]) \right].$$

Then the result follows by replacing all Lie brackets in the above formula by co-variant derivatives using  $[X, Y] = \nabla_X Y - \nabla_Y X$  and by expanding all terms of the form  $X((D_{(f,f_t)}g)(Y, Z))$  using

$$X((D_{(f,f_t)}g)(Y, Z)) = (\nabla_X D_{(f,f_t)}g)(Y, Z) + (D_{(f,f_t)}g)(\nabla_X Y, Z) + (D_{(f,f_t)}g)(Y, \nabla_X Z). \quad \square$$

**3.13. Setting for second variations.** All formulas for second derivatives will be used in section 7.2. There we consider a curve of immersions

$$f(t, x) = \exp_{f_0(x)}(t.a(x).\nu^{f_0}(x)) = f_0(x) + t.a(x).\nu^{f_0}(x),$$

for a fixed immersion  $f_0$ . This curve of immersions has the property that at  $t = 0$  its first derivative and the covariant derivative of the first derivative are both horizontal, i.e.,

$$(1) \quad f|_{t=0} = f_0, \quad \partial_t|_{t=0} f = a.\nu^{f_0}, \quad \text{and} \quad \nabla_{\partial_t} T f . \partial_t|_{t=0} = 0.$$

In all calculations of second variations we will assume that the above properties hold.

**3.14. Lemma** (Second variation of the metric). *The second derivative of the pull-back metric*

$$\begin{cases} \text{Imm} & \rightarrow \Gamma(S_{>0}^2 T^* M), \\ f & \mapsto g = f^* \bar{g} \end{cases}$$

along a curve of immersions  $f$  satisfying property (1) from section 3.13 is given by

$$\partial_t^2|_0 f^* \bar{g} = 2(da \otimes da) + 2a^2 g_0 \circ (L^{f_0} \otimes L^{f_0}).$$

*Proof.* Since  $\nabla_{\partial_t} T f . \partial_t|_0 = 0$ , we have

$$\begin{aligned} \partial_t^2|_0 g(X, Y) &= \partial_t^2|_0 \bar{g}(T f . X, T f . Y) \\ &= \partial_t|_0 \bar{g}(\nabla_{\partial_t} T f . X, T f . Y) + \partial_t|_0 \bar{g}(T f . X, \nabla_{\partial_t} T f . Y) \\ &= 2\bar{g}(\nabla_{\partial_t} T f . X|_0, \nabla_{\partial_t} T f . Y|_0) + 0 + 0 = 2\bar{g}(\nabla_X T f . \partial_t, \nabla_Y T f . \partial_t) \end{aligned}$$

Using  $T f . \partial_t = a.\nu^{f_0}$  we get

$$\begin{aligned} \partial_t^2|_0(g(X, Y)) &= 2da(X).da(Y) + 2a^2 \bar{g}(\nabla_X \nu^{f_0}, \nabla_Y \nu^{f_0}) \\ &= 2(da \otimes da)(X, Y) + 2a^2 .g_0(L^{f_0} X, L^{f_0} Y) \end{aligned} \quad \square$$

**3.15. Lemma** (Second variation of the inverse metric). *The second derivative of the inverse of the pullback metric*

$$\begin{cases} \text{Imm} & \rightarrow \Gamma(L(T^* M, TM)), \\ f & \mapsto g^{-1} = (f^* \bar{g})^{-1} \end{cases}$$

along a curve of immersions  $f$  satisfying property (1) from section 3.13 is given by

$$\partial_t^2|_0 (f^* \bar{g})^{-1} = 6a^2 (L^{f_0})^2 .g_0^{-1} - 2g_0^{-1} (da \otimes da) g_0^{-1}.$$

*Proof.* We look at  $g = f^*\bar{g}$  as a bundle map from  $TM$  to  $T^*M$ . Then

$$\begin{aligned}\partial_t^2|_0(g^{-1}) &= \partial_t|_0(-g^{-1} \cdot \partial_t g \cdot g^{-1}) = 2g_0^{-1} \cdot \partial_t|_0 g \cdot g_0^{-1} \cdot \partial_t|_0 g \cdot g_0^{-1} - g_0^{-1} \cdot \partial_t^2|_0 g \cdot g_0^{-1} \\ &= 2(-2aL^{f_0})^2 \cdot g_0^{-1} - g_0^{-1} \cdot (2(da \otimes da) + 2a^2 g_0 \circ (L^{f_0} \otimes L^{f_0})) \cdot g_0^{-1} \\ &= 8a^2(L^{f_0})^2 \cdot g_0^{-1} - 2g_0^{-1}(da \otimes da)g_0^{-1} - 2a^2(L^{f_0})^2 \cdot g_0^{-1} \quad \square\end{aligned}$$

**3.16. Lemma** (Second variation of the volume form). *The second derivative of the volume form*

$$\begin{cases} \text{Imm} & \rightarrow \Omega^{n-1}(M), \\ f & \mapsto \text{vol}(g) = \text{vol}(f^*\bar{g}) \end{cases}$$

along a curve of immersions  $f$  satisfying property (1) from section 3.13 is given by

$$\partial_t^2|_0 \text{vol}(g) = \left[ a^2 \text{Tr}(L^{f_0})^2 - a^2 \text{Tr}((L^{f_0})^2) + \|da\|_{g^{-1}}^2 \right] \text{vol}(g_0),$$

*Proof.* In section 3.6 we showed that for any curve of Riemannian metrics  $g(t) \in \Gamma(S_{>0}^2 T^*M)$ , we have

$$\partial_t \text{vol}(g) = \frac{1}{2} \text{Tr}(g^{-1} \cdot \partial_t g) \text{vol}(g).$$

Therefore

$$\begin{aligned}\partial_t^2 \text{vol}(g) &= \partial_t \frac{1}{2} \text{Tr}(g^{-1} \cdot \partial_t g) \text{vol}(g) = \frac{1}{2} \text{Tr}(\partial_t(g^{-1}) \cdot \partial_t g) \text{vol}(g) \\ &\quad + \frac{1}{2} \text{Tr}(g^{-1} \cdot \partial_t^2 g) \text{vol}(g) + \frac{1}{2} \text{Tr}(g^{-1} \cdot \partial_t g) \partial_t \text{vol}(g)\end{aligned}$$

Evaluating at  $t = 0$  and setting  $g(t) = f^*\bar{g}$  we get

$$\begin{aligned}\partial_t^2|_0 \text{vol}(g) &= \frac{1}{2} \text{Tr}((2aL^{f_0}g_0^{-1}) \cdot (-2a \cdot s^{f_0})) \text{vol}(g_0) \\ &\quad + \frac{1}{2} \text{Tr}(g_0^{-1} \cdot 2(da \otimes da)) \text{vol}(g_0) \\ &\quad + \frac{1}{2} \text{Tr}(g_0^{-1} \cdot 2a^2 g_0 \cdot (L^{f_0})^2) \text{vol}(g_0) \\ &\quad + \frac{1}{2} \text{Tr}(g_0^{-1} \cdot (-2a \cdot s^{f_0})) (-\text{Tr}(L^{f_0}) \cdot a) \text{vol}(g_0) \\ &= \left[ a^2 \text{Tr}(L^{f_0})^2 - a^2 \text{Tr}((L^{f_0})^2) + \|da\|_{g^{-1}}^2 \right] \text{vol}(g_0) \quad \square\end{aligned}$$

**3.17. Lemma** (Second variation of the second fundamental form). *The second derivative of the second fundamental form*

$$\begin{cases} \text{Imm} & \rightarrow \Gamma(S^2 T^*M), \\ f & \mapsto s^f \end{cases}$$

along a curve of immersions  $f$  satisfying Property (3.13.1) is given by

$$\begin{aligned}\partial_t^2|_0 s &= 2(da \otimes da)(\text{Id} \otimes L^{f_0} + L^{f_0} \otimes \text{Id}) - \|da\|_{g_0^{-1}}^2 \cdot s^{f_0} \\ &\quad + 2a(\nabla_{\text{grad}^{g_0}(a)} s^{f_0}).\end{aligned}$$

*Proof.* From section 3.8 we have

$$\partial_t s(X, Y) = \bar{g}(\bar{\nabla}_{X,Y}^2 T f \cdot \partial_t, \nu) = \bar{g}(\bar{\nabla}_{X,Y}^2 f_t, \nu).$$



Using  $\bar{\nabla}_{\partial_t} f_t = 0$  we get

$$\begin{aligned}\partial_t^2 s(X, Y) &= \bar{g}(\bar{\nabla}_{X,Y}^2 f_t, \bar{\nabla}_{\partial_t} \nu) + \bar{g}(\bar{\nabla}_{\partial_t} \bar{\nabla}_X \bar{\nabla}_Y f_t - \bar{\nabla}_{\partial_t} \bar{\nabla}_{\nabla_X Y} f_t, \nu) \\ &= \bar{g}(\bar{\nabla}_{X,Y}^2 f_t, \bar{\nabla}_{\partial_t} \nu) + 0 - \bar{g}(\bar{\nabla}_{\nabla_X Y} \bar{\nabla}_{\partial_t} f_t + \bar{\nabla}_{[\partial_t, \nabla_X Y]} f_t, \nu) \\ &= \bar{g}(\bar{\nabla}_{X,Y}^2 f_t, \bar{\nabla}_{\partial_t} \nu) + 0 - \bar{g}(\bar{\nabla}_{[\partial_t, \nabla_X Y]} f_t, \nu) \\ &= \bar{g}(\bar{\nabla}_{X,Y}^2 f_t, \bar{\nabla}_{\partial_t} \nu) - \bar{g}(\bar{\nabla}_{(D_{(f,f_t)} \nabla)(X,Y)} f_t, \nu)\end{aligned}$$

In the last step we used

$$\begin{aligned}[\partial_t, \nabla_X^{f*} \bar{g} Y] &= [(\partial_t, 0_M), (0_{\mathbb{R}}, \nabla_X^{f*} \bar{g} Y)] \\ &= (0_{\mathbb{R}}, (D_{(f,f_t)} \nabla)(X, Y)) = (D_{(f,f_t)} \nabla)(X, Y).\end{aligned}$$

Evaluating at  $t = 0$  yields:

$$\begin{aligned}\partial_t^2|_0 s(X, Y) &= \\ &= \bar{g}(\bar{\nabla}_{X,Y}^2(a.\nu^{f_0}), -T f_0.\text{grad}^{g_0} a) - \bar{g}(\bar{\nabla}_{(D_{(f_0,a.\nu^{f_0})} \nabla)(X,Y)}(a.\nu^{f_0}), \nu^{f_0}) \\ &= 0 + \bar{g}(da(X).\bar{\nabla}_Y \nu^{f_0} + da(Y).\bar{\nabla}_X \nu^{f_0}, -T f_0.\text{grad}^{g_0} a) \\ &\quad + \bar{g}(a.\bar{\nabla}_{X,Y}^2(\nu^{f_0}), -T f_0.\text{grad}^{g_0} a) - da((D_{(f_0,a.\nu^{f_0})} \nabla)(X, Y)) + 0.\end{aligned}$$

We will treat the three terms separately. The first one gives, using  $\bar{\nabla}_Z \nu = -T f.L.Z$ :

$$\begin{aligned}\bar{g}(da(X).\bar{\nabla}_Y \nu^{f_0} + da(Y).\bar{\nabla}_X \nu^{f_0}, -T f_0.\text{grad}^{g_0} a) &= \\ &= g_0(da(X)L^{f_0} Y + da(Y)L^{f_0} X, \text{grad}^{g_0} a) \\ &= da(X).da(L^{f_0} Y) + da(Y).da(L^{f_0} X).\end{aligned}$$

For the second term we get:

$$\begin{aligned}\bar{g}(a.\bar{\nabla}_{X,Y}^2(\nu^{f_0}), -T f_0.\text{grad}^{g_0} a) &= -a\bar{g}(\bar{\nabla}_X \bar{\nabla}_Y \nu^{f_0} - \bar{\nabla}_{\nabla_X Y} \nu^{f_0}, T f_0.\text{grad}^{g_0} a) \\ &= -a\bar{g}(-\bar{\nabla}_X(T f_0 L^{f_0} Y) + T f_0 L^{f_0} \nabla_X Y, T f_0.\text{grad}^{g_0} a) \\ &= -a\bar{g}(-(\bar{\nabla} T f_0)(X, L^{f_0} Y) - T f_0 \nabla_X(L^{f_0} Y) + T f_0 L^{f_0} \nabla_X Y, T f_0.\text{grad}^{g_0} a) \\ &= 0 + a\bar{g}(T f_0(\nabla_X L^{f_0})(Y), T f_0.\text{grad}^{g_0} a) = ag_0((\nabla_X L^{f_0})(Y), \text{grad}^{g_0} a) \\ &= a\nabla_X(g_0(L^{f_0} Y, \text{grad}^{g_0} a)) - ag_0(L^{f_0} \nabla_X Y, \text{grad}^{g_0} a) \\ &\quad - ag_0(L^{f_0} Y, \nabla_X \text{grad}^{g_0} a) \\ &= a\nabla_X(s^{f_0}(Y, \text{grad}^{g_0} a)) - as^{f_0}(\nabla_X Y, \text{grad}^{g_0} a) - as^{f_0}(Y, \nabla_X \text{grad}^{g_0} a) \\ &= a(\nabla_X s)(Y, \text{grad}^{g_0} a)\end{aligned}$$

$\bar{\nabla}_{X,Y}^2 \nu$  is symmetric in  $X, Y$  because the ambient space  $\mathbb{R}^n$  is flat. Therefore the last formula and the symmetry of  $s$  imply that

$$a(\nabla_X s)(Y, \text{grad}^{g_0} a) = a(\nabla_Y s)(X, \text{grad}^{g_0} a) = a(\nabla_{\text{grad}^{g_0} a} s)(X, Y).$$

The third term yields, using formula 3.12:

$$\begin{aligned}-g_0((D_{(f,a.\nu^{f_0})} \nabla)(X, Y), \text{grad}^{g_0}(a)) &= \\ &= -\frac{1}{2}(\nabla(-2a.s^{f_0}))(X, Y, \text{grad}^{g_0}(a)) - \frac{1}{2}(\nabla(-2a.s^{f_0}))(Y, X, \text{grad}^{g_0}(a)) \\ &\quad + \frac{1}{2}(\nabla(-2a.s^{f_0}))(\text{grad}^{g_0}(a), X, Y)\end{aligned}$$

$$\begin{aligned}
&= da(X).s^{f_0}(Y, \text{grad}^{g_0}(a)) + a.(\nabla_X s^{f_0})(Y, \text{grad}^{g_0}(a)) \\
&\quad + da(Y).s^{f_0}(X, \text{grad}^{g_0}(a)) + a.(\nabla_Y s^{f_0})(X, \text{grad}^{g_0}(a)) \\
&\quad - da(\text{grad}^{g_0}(a)).s^{f_0}(X, Y) - a.(\nabla_{\text{grad}^{g_0}(a)} s^{f_0})(X, Y) \\
&= da(X).da(L^{f_0}Y) + da(Y).da(L^{f_0}X) \\
&\quad - \|da\|_{g^{-1}}^2.s^{f_0}(X, Y) + a.(\nabla_{\text{grad}^{g_0}(a)} s^{f_0})(X, Y). \quad \square
\end{aligned}$$

**3.18. Lemma** (Second variation of the mean curvature). *The second derivative of the mean curvature*

$$\begin{cases} \text{Imm} & \rightarrow C^\infty(M), \\ f & \mapsto \text{Tr}(L^f) \end{cases}$$

along a curve of immersions  $f$  satisfying Property (3.13.1) is given by

$$\begin{aligned}
\partial_t^2|_0 \text{Tr}(L) &= 2a^2 \text{Tr}((L^{f_0})^3) + 4a \text{Tr}(L^{f_0}g_0^{-1}.\nabla^2 a) + 2 \text{Tr}(g^{-1}(da \otimes da)L^{f_0}) \\
&\quad - \|da\|_{g_0^{-1}}^2 \text{Tr}(L^{f_0}) + 2a \text{Tr}^{g_0}(\nabla_{\text{grad}^{g_0} a} s^{f_0})
\end{aligned}$$

*Proof.* From  $\text{Tr}(L) = \text{Tr}(g^{-1}.s)$  we get

$$\partial_t^2 \text{Tr}(L) = \text{Tr}(\partial_t^2(g^{-1}).s) + 2 \text{Tr}(\partial_t(g^{-1}).\partial_t s) + \text{Tr}(g^{-1}.\partial_t^2 s)$$

Evaluating at  $t = 0$  we get

$$\begin{aligned}
\partial_t^2|_0 \text{Tr}(L) &= \text{Tr}(6a^2(L^{f_0})^2.g_0^{-1}.s^{f_0}) + \text{Tr}(-2g_0^{-1}.(da \otimes da).g_0^{-1}.s^{f_0}) \\
&\quad + 2 \text{Tr}(2aL^{f_0}g_0^{-1}.\nabla^2 a) + 2 \text{Tr}(2aL^{f_0}g_0^{-1}.(-ag_0(L^{f_0})^2)) \\
&\quad + 2 \text{Tr}(g_0^{-1}((da \otimes da \circ L^{f_0}) + (da \circ L^{f_0} \otimes da)) \\
&\quad - \|da\|_{g_0^{-1}}^2 \text{Tr}(L^{f_0}) + 2a \text{Tr}^{g_0}(\nabla_{\text{grad}^{g_0} a} s^{f_0}) \\
&= 2a^2 \text{Tr}((L^{f_0})^3) - 2 \text{Tr}(g_0^{-1}.(da \otimes da).L^{f_0}) \\
&\quad + 4a \text{Tr}(L^{f_0}g_0^{-1}.\nabla^2 a) + 4 \text{Tr}(g_0^{-1}.(da \otimes da).(L^{f_0})) \\
&\quad - \|da\|_{g_0^{-1}}^2 \text{Tr}(L^{f_0}) + 2a \text{Tr}^{g_0}(\nabla_{\text{grad}^{g_0} a} s^{f_0}) \quad \square
\end{aligned}$$

#### 4. THE HAMILTONIAN APPROACH

See [13, section 2] for a detailed exposition in similar notation.

**4.1. The setting.** Consider the smooth Fréchet manifold  $\text{Imm}(M, \mathbb{R}^n)$  of all immersions  $M \rightarrow \mathbb{R}^n$ , which is an open subset of  $C^\infty(M, \mathbb{R}^n)$ .

The tangent bundle is  $T \text{Imm}(M, \mathbb{R}^n) = \text{Imm}(M, \mathbb{R}^n) \times C^\infty(M, \mathbb{R}^n)$ , and the cotangent bundle is  $T^* \text{Imm}(M, \mathbb{R}^n) = \text{Imm}(M, \mathbb{R}^n) \times \mathcal{D}(M)^n$  where the second factor consists of  $n$ -tuples of distributions in  $\mathcal{D}(M) = C^\infty(M)'$  which is the space of distributional sections of the density bundle.

We consider smooth Riemannian metrics on  $\text{Imm}(M, \mathbb{R}^n)$ , i.e., smooth mappings

$$\begin{aligned}
G &: \text{Imm}(M, \mathbb{R}^n) \times C^\infty(M, \mathbb{R}^n) \times C^\infty(M, \mathbb{R}^n) \rightarrow \mathbb{R} \\
(f, h, k) &\mapsto G_f(h, k), \quad \text{bilinear in } h, k
\end{aligned}$$

$$G_f(h, h) > 0 \quad \text{for } h \neq 0.$$

Each such metric is *weak* in the sense that  $G_f$ , viewed as bounded linear mapping

$$\begin{aligned} G_f : T_f \text{Imm}(M, \mathbb{R}^n) &= C^\infty(M, \mathbb{R}^n) \rightarrow T_f^* \text{Imm}(M, \mathbb{R}^n) = \mathcal{D}(M)^n \\ G : T \text{Imm}(M, \mathbb{R}^n) &\rightarrow T^* \text{Imm}(M, \mathbb{R}^n) \\ G(f, h) &= (f, G_f(h, \cdot)) \end{aligned}$$

is injective, but can never be surjective. We shall need also its tangent mapping

$$TG : T(T \text{Imm}(M, \mathbb{R}^n)) \rightarrow T(T^* \text{Imm}(M, \mathbb{R}^n))$$

We write a tangent vector to  $T \text{Imm}(M, \mathbb{R}^n)$  in the form  $(f, h; k, V)$  where  $(f, h) \in T \text{Imm}(M, \mathbb{R}^n)$  is its foot point,  $k$  is its vector component in the  $\text{Imm}(M, \mathbb{R}^n)$ -direction and where  $V$  is its component in the  $C^\infty(M, \mathbb{R}^n)$ -direction.

Then  $TG$  is given by

$$TG(f, h; k, V) = (f, G_f(h, \cdot); k, D_{(f,k)}G_f(h, \cdot) + G_f(V, \cdot))$$

Moreover, if  $X = (f, h; k, V)$  then we will write  $X_1 = k$  for its first vector component and  $X_2 = V$  for the second vector component.

Note that only these smooth functions on  $\text{Imm}(M, \mathbb{R}^n)$  whose derivative lies in the image of  $G$  in the cotangent bundle have  $G$ -gradients. This requirement has only to be satisfied for the first derivative, for the higher ones it follows (see [11]). We shall denote by  $C_G^\infty(\text{Imm}(M, \mathbb{R}^n))$  the space of such smooth functions.

We shall always assume that  $G$  is invariant under the reparametrization group  $\text{Diff}(M)$ , hence each such metric induces a Riemann-metric on the quotient space  $B_i(M, \mathbb{R}^n) = \text{Imm}(M, \mathbb{R}^n) / \text{Diff}(M)$ .

In the sequel we shall further assume that *the weak Riemannian metric  $G$  itself admits  $G$ -gradients with respect to the variable  $f$  in the following sense:*

$$\boxed{D_{(f,m)}G_f(h, k) = G_f(m, H_f(h, k)) = G_f(K_f(m, h), k)} \quad \text{where} \\ H, K : \text{Imm}(M, \mathbb{R}^n) \times C^\infty(M, \mathbb{R}^n) \times C^\infty(M, \mathbb{R}^n) \rightarrow C^\infty(M, \mathbb{R}^n) \\ (f, h, k) \mapsto H_f(h, k), K_f(h, k)$$

are smooth and bilinear in  $h, k$ .

Note that  $H$  and  $K$  could be expressed in abstract index notation as  $g_{ij,k}g^{kl}$  and  $g_{ij,k}g^{il}$ . We will check and compute these gradients for several concrete metrics below.

**4.2. The fundamental symplectic form on  $T \text{Imm}(M, \mathbb{R}^n)$  induced by a weak Riemannian metric.** The basis of Hamiltonian theory is the natural 1-form on the cotangent bundle  $T^* \text{Imm}(M, \mathbb{R}^n)$  given by:

$$\begin{aligned} \Theta : T(T^* \text{Imm}(M, \mathbb{R}^n)) &= \text{Imm}(M, \mathbb{R}^n) \times \mathcal{D}(M)^n \times C^\infty(M, \mathbb{R}^n) \times \mathcal{D}(M)^n \rightarrow \mathbb{R} \\ (f, \alpha; h, \beta) &\mapsto \langle \alpha, h \rangle. \end{aligned}$$

The pullback via the mapping  $G : T \text{Imm}(M, \mathbb{R}^n) \rightarrow T^* \text{Imm}(M, \mathbb{R}^n)$  of the 1-form  $\Theta$  is then:

$$(G^* \Theta)_{(f,h)}(f, h; k, V) = G_f(h, k).$$

Thus the symplectic form  $\omega = -dG^*\Theta$  on  $T\text{Imm}(M, \mathbb{R}^n)$  can be computed as follows, where we use the constant vector fields  $(f, h) \mapsto (f, h; k, V)$ :

$$\begin{aligned}
 \omega_{(f,h)}((k_1, V_1), (k_2, V_2)) &= -d(G^*\Theta)((k_1, V_1), (k_2, V_2))|_{(f,h)} \\
 &= -D_{(f,k_1)}G_f(h, k_2) - G_f(V_1, k_2) + D_{(f,k_2)}G_f(h, k_1) + G_f(V_2, k_1) \\
 (1) \quad &= G_f(k_2, H_f(h, k_1) - K_f(k_1, h)) + G_f(V_2, k_1) - G_f(V_1, k_2)
 \end{aligned}$$

**4.3. The Hamiltonian vector field mapping.** Here we compute the Hamiltonian vector field  $\text{grad}^\omega(F)$  associated to a smooth function  $F$  on the tangent space  $T\text{Imm}(M, \mathbb{R}^n)$ , that is  $F \in C_G^\infty(\text{Imm}(M, \mathbb{R}^n) \times C^\infty(M, \mathbb{R}^n))$  assuming that it has smooth  $G$ -gradients in both factors. See [11, section 48].

Using the explicit formulas in 4.2, we have:

$$\begin{aligned}
 \omega_{(f,h)}(\text{grad}^\omega(F)(f, h), (k, V)) &= \omega_{(f,h)}((\text{grad}_1^\omega(F)(f, h), \text{grad}_2^\omega(F)(f, h)), (k, V)) = \\
 &= G_f(k, H_f(h, \text{grad}_1^\omega(F)(f, h))) - G_f(K_f(\text{grad}_1^\omega(F)(f, h), h), k) \\
 &\quad + G_f(V, \text{grad}_1^\omega(F)(f, h)) - G_f(\text{grad}_2^\omega(F)(f, h), k)
 \end{aligned}$$

On the other hand, by the definition of the  $\omega$ -gradient we have

$$\begin{aligned}
 \omega_{(f,h)}(\text{grad}^\omega(F)(f, h), (k, V)) &= dF(f, h)(k, V) = D_{(f,k)}F(f, h) + D_{(h,V)}F(f, h) \\
 &= G_f(\text{grad}_1^G(F)(f, h), k) + G_f(\text{grad}_2^G(F)(f, h), V)
 \end{aligned}$$

and we get the expression of the Hamiltonian vector field:

$$\begin{aligned}
 \text{grad}_1^\omega(F)(f, h) &= \text{grad}_2^G(F)(f, h) \\
 \text{grad}_2^\omega(F)(f, h) &= -\text{grad}_1^G(F)(f, h) \\
 &\quad + H_f(h, \text{grad}_2^G(F)(f, h)) - K_f(\text{grad}_2^G(F)(f, h), h)
 \end{aligned}$$

Note that for a smooth function  $F$  on  $T\text{Imm}(M, \mathbb{R}^n)$  the  $\omega$ -gradient exists if and only if both  $G$ -gradients exist.

**4.4. The geodesic equation.** The geodesic flow is defined by a vector field on  $T\text{Imm}(M, \mathbb{R}^n)$ . One way to define this vector field is as the Hamiltonian vector field of the energy function

$$E(f, h) = \frac{1}{2}G_f(h, h), \quad E : \text{Imm}(M, \mathbb{R}^n) \times C^\infty(M, \mathbb{R}^n) \rightarrow \mathbb{R}.$$

The two partial  $G$ -gradients are:

$$\begin{aligned}
 G_f(\text{grad}_2^G(E)(f, h), V) &= d_2E(f, h)(V) = G_f(h, V) \\
 \text{grad}_2^G(E)(f, h) &= h \\
 G_f(\text{grad}_1^G(E)(f, h), k) &= d_1E(f, h)(k) = \frac{1}{2}D_{(f,k)}G_f(h, h) \\
 &= \frac{1}{2}G_f(k, H_f(h, h)) \\
 \text{grad}_1^G(E)(f, h) &= \frac{1}{2}H_f(h, h).
 \end{aligned}$$

Thus the geodesic vector field is

$$\text{grad}_1^\omega(E)(f, h) = h$$

$$\text{grad}_2^\omega(E)(f, h) = \frac{1}{2}H_f(h, h) - K_f(h, h)$$

and the geodesic equation becomes:

$$\begin{cases} f_t &= h \\ h_t &= \frac{1}{2}H_f(h, h) - K_f(h, h) \end{cases} \quad \text{or} \quad \boxed{f_{tt} = \frac{1}{2}H_f(f_t, f_t) - K_f(f_t, f_t)}$$

This is nothing but the usual formula for the geodesic flow using the Christoffel symbols expanded out using the first derivatives of the metric tensor.

**4.5. The momentum mapping for a  $G$ -isometric group action.** We consider now a (possibly infinite dimensional regular) Lie group with Lie algebra  $\mathfrak{g}$  with a right action  $g \mapsto r^g$  by isometries on  $\text{Imm}(M, \mathbb{R}^n)$ . Denote by  $\mathfrak{X}(\text{Imm}(M, \mathbb{R}^n))$  the set of vector fields on  $\text{Imm}(M, \mathbb{R}^n)$ . Then we can specify this action by the fundamental vector field mapping  $\zeta : \mathfrak{g} \rightarrow \mathfrak{X}(\text{Imm}(M, \mathbb{R}^n))$ , which will be a bounded Lie algebra homomorphism. The fundamental vector field  $\zeta_X, X \in \mathfrak{g}$  is the infinitesimal action in the sense:

$$\zeta_X(f) = \partial_t|_0 r^{\exp(tX)}(f).$$

We also consider the tangent prolongation of this action on  $T\text{Imm}(M, \mathbb{R}^n)$  where the fundamental vector field is given by

$$\zeta_X^{T\text{Imm}} : (f, h) \mapsto (f, h; \zeta_X(f), D_{(f,h)}(\zeta_X)(f) =: \zeta'_X(f, h))$$

The basic assumption is that the action is by isometries,

$$G_f(h, k) = ((r^g)^*G)_f(h, k) = G_{r^g(f)}(T_f(r^g)h, T_f(r^g)k).$$

Differentiating this equation at  $g = e$  in the direction  $X \in \mathfrak{g}$  we get

$$(1) \quad 0 = D_{(f, \zeta_X(f))}G_f(h, k) + G_f(\zeta'_X(f, h), k) + G_f(h, \zeta'_X(f, k))$$

The key to the Hamiltonian approach is to define the group action by Hamiltonian flows. We define the *momentum map*  $j : \mathfrak{g} \rightarrow C_G^\infty(T\text{Imm}(M, \mathbb{R}^n), \mathbb{R})$  by:

$$\boxed{j_X(f, h) = G_f(\zeta_X(f), h).}$$

Equivalently, since this map is linear, it is often written as a map

$$\mathcal{J} : T\text{Imm}(M, \mathbb{R}^n) \rightarrow \mathfrak{g}', \quad \langle \mathcal{J}(f, h), X \rangle = j_X(f, h).$$

The main property of the momentum map is that it fits into the following commutative diagram and is a homomorphism of Lie algebras:

$$\begin{array}{ccccc} H^0(T\text{Imm}) & \xrightarrow{i} & C_G^\infty(T\text{Imm}, \mathbb{R}) & \xrightarrow{\text{grad}^\omega} & \mathfrak{X}(T\text{Imm}, \omega) \longrightarrow H^1(T\text{Imm}) \\ & & \nwarrow j & \nearrow \zeta^{T\text{Imm}} & \\ & & \mathfrak{g} & & \end{array}$$

where  $\mathfrak{X}(T\text{Imm}, \omega)$  is the space of vector fields on  $T\text{Imm}$  whose flow leaves  $\omega$  fixed. Note also that  $\mathcal{J}$  is equivariant for the group action. See [17] for more details.

By Emmy Noether's theorem, along any geodesic  $t \mapsto c(t, \cdot)$  this momentum mapping is constant, thus for any  $X \in \mathfrak{g}$  we have

$$\boxed{\langle \mathcal{J}(f, f_t), X \rangle = j_X(f, f_t) = G_f(\zeta_X(f), f_t) \quad \text{is constant in } t.}$$

We can apply this construction to the following group actions:

- The smooth right action of the group  $\text{Diff}(M)$  on  $\text{Imm}(M, \mathbb{R}^n)$ , given by composition from the right:  $f \mapsto f \circ \varphi$  for  $\varphi \in \text{Diff}(M)$ .

For  $X \in \mathfrak{X}(M)$  the fundamental vector field is then given by

$$\zeta_X^{\text{Diff}}(f) = \zeta_X(f) = X(f) = \partial_t|_0(f \circ \text{Fl}_t^X) = df.X$$

where  $\text{Fl}_t^X$  denotes the flow of  $X$ . The *reparametrization momentum*, for any vector field  $X$  on  $M$  is thus:

$$j_X(f, h) = G_f(df.X, h).$$

Assuming the metric is reparametrization invariant, it follows that on any geodesic  $f(x, t)$ , the expression  $G_f(df.X, f_t)$  is constant for all  $X$ .

- The left action of the Euclidean motion group  $\mathbb{R}^n \rtimes SO(n)$  on  $\text{Imm}(M, \mathbb{R}^n)$  given by  $f \mapsto Af + B$  for  $(B, A) \in \mathbb{R}^n \times SO(n)$ . The fundamental vector field mapping is

$$\zeta_{(B, X)}(f) = Xf + B$$

The *linear momentum* is thus  $G_f(B, h)$ ,  $B \in \mathbb{R}^n$  and if the metric is translation invariant,  $G_f(B, f_t)$  will be constant along geodesics for every  $B \in \mathbb{R}^n$ . The *angular momentum* is similarly  $G_f(X.f, h)$ ,  $X \in \mathfrak{so}(n)$  and if the metric is rotation invariant, then  $G_f(X.f, f_t)$  will be constant along geodesics for each  $X \in \mathfrak{so}(n)$ .

- The action of the scaling group of  $\mathbb{R}$  given by  $c \mapsto e^r f$ , with fundamental vector field  $\zeta_a(f) = a.f$ .

If the metric is scale invariant, then the *scaling momentum*  $G_f(f, f_t)$  will also be invariant along geodesics.

## 5. THE GEODESIC EQUATION ON $\text{Imm}(M, \mathbb{R}^n)$

The  $G^0$  metric is the simplest metric on  $\text{Imm}(M, \mathbb{R}^n)$ . It is given by the following formula:

$$G_f^0(h, k) = \int_M \bar{g}(h, k) \text{vol}(g).$$

This metric is well studied, see for example [15]. In this chapter we will study metrics of the form

$$G_f^\Phi(h, k) = \int_M \Phi(f) \bar{g}(h(x), k(x)) \text{vol}(g)(x),$$

where  $\Phi$  is a  $\text{Diff}(M)$ -invariant function depending on the immersion  $f$  and possibly on  $x$ . These metrics are called *almost local* metrics. This definition includes as an important special case *conformal versions* of the  $G^0$  metric, i.e. metrics of the form

$$G_f^\Phi(h, k) = \Phi(f) \int_M \bar{g}(h, k) \text{vol}(g),$$

where  $\Phi$  is again some  $\text{Diff}(M)$ -invariant function depending on the immersion  $f$  but not on  $x$ . Conformal metrics have been studied in [19, 12]. We will consider functions  $\Phi$  depending on the volume and mean curvature, i.e.

$$\Phi = \Phi(\text{Vol}, \text{Tr}(L^f)(x)).$$

For this class of functions, some work has already been done by [17] for the special case of immersions of the unit circle in the plane.

**5.1. The geodesic equation on  $\text{Imm}(M, \mathbb{R}^n)$ .** We use the method of section 4.4 to calculate the geodesic equation. So we need to compute the metric gradients. The calculation at the same time shows the existence of the gradients. Let  $m \in T_f \text{Imm}(M, \mathbb{R}^n)$  with

$$m = a.\nu^f + T f.m^\top.$$

To shorten the notation, we will not always note the dependence on  $f$  in expressions as  $\nu^f, L^f \dots$

$$\begin{aligned} D_{(f,m)} G_f^\Phi(h, k) &= \int_M (\partial_1 \Phi)(D_{(f,m)} \text{Vol}) \bar{g}(h, k) \text{vol}(g) \\ &\quad + \int_M (\partial_2 \Phi)(D_{(f,m)} \text{Tr}(L)) \bar{g}(h, k) \text{vol}(g) \\ &\quad + \int_M \Phi.\bar{g}(h, k)(D_{(f,m)} \text{vol}(g)). \end{aligned}$$

To read off the  $K$ -gradient of the metric, we write this expression as

$$\int_M \Phi.\bar{g} \left( \left[ \frac{\partial_1 \Phi}{\Phi}(D_{(f,m)} \text{Vol}) + \frac{\partial_2 \Phi}{\Phi}(D_{(f,m)} \text{Tr}(L)) + \frac{D_{(f,m)} \text{vol}(g)}{\text{vol}(g)} \right] h, k \right) \text{vol}(g)$$

Therefore, using the formulas from section 3 we can calculate the  $K$  gradient:

$$\begin{aligned} K_f(m, h) &= \left[ \frac{\partial_1 \Phi}{\Phi}(D_{(f,m)} \text{Vol}) + \frac{\partial_2 \Phi}{\Phi}(D_{(f,m)} \text{Tr}(L)) + \frac{D_{(f,m)} \text{vol}(g)}{\text{vol}(g)} \right] h \\ &= \left[ \frac{\partial_1 \Phi}{\Phi} \left( \int_M -\text{Tr}(L).a \text{vol}(g) \right) \right. \\ &\quad \left. + \frac{\partial_2 \Phi}{\Phi} (-\Delta a + a \text{Tr}(L^2) + d \text{Tr}(L)(m^\top)) \right. \\ &\quad \left. + \text{div}^g(m^\top) - \text{Tr}(L).a \right] h. \end{aligned}$$

To calculate the  $H$ -gradient, we treat the three summands of  $D_{(f,m)} G_f^\Phi(h, k)$  separately. The first summand is

$$\begin{aligned} &\int_M (\partial_1 \Phi)(D_{(f,m)} \text{Vol}(x)) \bar{g}(h(x), k(x)) \text{vol}(g)(x) \\ &= \int_{x \in M} (\partial_1 \Phi) \left( \int_{y \in M} -\text{Tr}(L(y)).a(y) \text{vol}(g)(y) \right) \bar{g}(h(x), k(x)) \text{vol}(g)(x) \\ &= \int_{y \in M} \bar{g} \left( a(y).\nu(y), -\text{Tr}(L(y)) \int_{x \in M} (\partial_1 \Phi) \bar{g}(h(x), k(x)) \text{vol}(g)(x).\nu(y) \right) \text{vol}(g)(y) \\ &= G_f^\Phi \left( m, -\frac{1}{\Phi} \text{Tr}(L) \int_M (\partial_1 \Phi) \bar{g}(h, k) \text{vol}(g).\nu \right). \end{aligned}$$

We will use the following formula, which is valid for any two functions  $a, b \in C^\infty(M)$ :

$$\int_M \Delta(a).b \text{vol}(g) = \int_M a.\Delta(b) \text{vol}(g).$$

Thus the second summand is given by

$$\int_M (\partial_2 \Phi)(D_{(f,m)} \text{Tr}(L)) \bar{g}(h, k) \text{vol}(g)$$

$$\begin{aligned}
&= \int_M (\partial_2 \Phi) (-\Delta a + a \operatorname{Tr}(L^2) + d \operatorname{Tr}(L)(m^\top)) \bar{g}(h, k) \operatorname{vol}(g) \\
&= \int_M -a \Delta((\partial_2 \Phi) \bar{g}(h, k)) \operatorname{vol}(g) + \int_M a \cdot (\partial_2 \Phi) \operatorname{Tr}(L^2) \bar{g}(h, k) \operatorname{vol}(g) \\
&\quad + \int_M (\partial_2 \Phi) g(\operatorname{grad}^g(\operatorname{Tr}(L)), m^\top) \bar{g}(h, k) \operatorname{vol}(g) \\
&= \int_M -a \Delta((\partial_2 \Phi) \bar{g}(h, k)) \operatorname{vol}(g) + \int_M a \cdot (\partial_2 \Phi) \operatorname{Tr}(L^2) \bar{g}(h, k) \operatorname{vol}(g) \\
&\quad + \int_M (\partial_2 \Phi) \bar{g}(Tf \cdot \operatorname{grad}^g(\operatorname{Tr}(L)), Tf \cdot m^\top) \bar{g}(h, k) \operatorname{vol}(g) \\
&= G_f^\Phi \left( m, -\frac{1}{\Phi} \Delta((\partial_2 \Phi) \bar{g}(h, k)) \cdot \nu \right) + G_f^\Phi \left( m, \frac{1}{\Phi} (\partial_2 \Phi) \operatorname{Tr}(L^2) \bar{g}(h, k) \cdot \nu \right) \\
&\quad + G_f^\Phi \left( m, \frac{1}{\Phi} (\partial_2 \Phi) \bar{g}(h, k) Tf \cdot \operatorname{grad}^g(\operatorname{Tr}(L)) \right)
\end{aligned}$$

In the calculation of the third term of the  $H_f(m, h)$  gradient, we will make use of the following formula, which is valid for  $\phi \in C^\infty(M)$  and  $X \in \mathfrak{X}(M)$ :

$$\begin{aligned}
0 &= \int_M \operatorname{div}(\phi \cdot X) \cdot \operatorname{vol}(g) = \int_M \mathcal{L}_{\phi \cdot X} \operatorname{vol}(g) \\
&= \int_M (d \circ i_{\phi \cdot X} + i_{\phi \cdot X} \circ d) \operatorname{vol}(g) = \int_M d(\phi \cdot i_X \operatorname{vol}(g)) \\
&= \int_M d\phi \wedge i_X \operatorname{vol}(g) + \int_M \phi \wedge d(i_X \operatorname{vol}(g)) \\
&= \int_M (-i_X(d\phi \wedge \operatorname{vol}(g)) + i_X \circ d\phi \wedge \operatorname{vol}(g)) + \int_M \phi \cdot \mathcal{L}_X \operatorname{vol}(g) \\
&= \int_M d\phi(X) \operatorname{vol}(g) + \int_M \phi \cdot \operatorname{div}(X) \operatorname{vol}(g).
\end{aligned}$$

Therefore we can calculate the third summand, which is given by

$$\begin{aligned}
&\int_M \Phi \bar{g}(h, k) (D_{(f, m)} \operatorname{vol}(g)) = \int_M \Phi \bar{g}(h, k) (\operatorname{div}^g(m^\top) - \operatorname{Tr}(L) \cdot a) \operatorname{vol}(g) \\
&= - \int_M d(\Phi \bar{g}(h, k))(m^\top) \operatorname{vol}(g) + G_f^\Phi(m, -\bar{g}(h, k) \operatorname{Tr}(L) \cdot \nu) \\
&= - \int_M \bar{g}(Tf \cdot \operatorname{grad}^g(\Phi \bar{g}(h, k)), m) \operatorname{vol}(g) + G_f^\Phi(m, -\bar{g}(h, k) \operatorname{Tr}(L) \cdot \nu) \\
&= G_f^\Phi \left( m, -\frac{1}{\Phi} Tf \cdot \operatorname{grad}^g(\Phi \bar{g}(h, k)) - \bar{g}(h, k) \operatorname{Tr}(L) \cdot \nu \right)
\end{aligned}$$

Summing up all the terms the  $H$ -gradient is given by

$$\begin{aligned}
H_f(h, k) &= \left[ -\frac{1}{\Phi} \operatorname{Tr}(L) \int_M (\partial_1 \Phi) \bar{g}(h, k) \operatorname{vol}(g) - \frac{1}{\Phi} \Delta((\partial_2 \Phi) \bar{g}(h, k)) \right. \\
&\quad + \frac{1}{\Phi} (\partial_2 \Phi) \operatorname{Tr}(L^2) \bar{g}(h, k) - \bar{g}(h, k) \operatorname{Tr}(L) \left. \right] \nu^f \\
&\quad + \frac{1}{\Phi} Tf \cdot \left[ (\partial_2 \Phi) \bar{g}(h, k) \operatorname{grad}^g(\operatorname{Tr}(L)) - \operatorname{grad}^g(\Phi \bar{g}(h, k)) \right].
\end{aligned}$$



**Theorem.** *The geodesic equation for the almost local metrics  $G^\Phi$  on  $\text{Imm}(M, \mathbb{R}^n)$  is then given by*

$$\begin{aligned}
 f_t &= h = a \cdot \nu^f + T f \cdot h^\top, \\
 h_t &= \frac{1}{2} \left[ -\frac{1}{\Phi} \text{Tr}(L) \int_M \partial_1 \Phi \|h\|^2 \text{vol}(g) - \frac{1}{\Phi} \Delta((\partial_2 \Phi) \|h\|^2) \right. \\
 &\quad \left. + \frac{1}{\Phi} (\partial_2 \Phi) \text{Tr}(L^2) \|h\|^2 - \|h\|^2 \text{Tr}(L) \right] \nu^f \\
 &\quad + \frac{1}{2\Phi} T f \cdot \left[ (\partial_2 \Phi) \|h\|^2 \text{grad}^g(\text{Tr}(L)) - \text{grad}^g(\Phi \|h\|^2) \right] \\
 &\quad - \left[ \frac{\partial_1 \Phi}{\Phi} \int_M -\text{Tr}(L) \cdot a \text{vol}(g) \right. \\
 &\quad \left. + \frac{\partial_2 \Phi}{\Phi} (-\Delta a + a \text{Tr}(L^2) + d \text{Tr}(L)(h^\top)) \right. \\
 &\quad \left. + \text{div}^g(h^\top) - \text{Tr}(L) \cdot a \right] h.
 \end{aligned}$$

**5.2. Momentum mappings.** The metric  $G^\Phi$  is invariant under the action of the reparametrization group  $\text{Diff}(M)$  and under the Euclidean motion group  $\mathbb{R}^n \rtimes \text{SO}(n)$ . According to section 4.5 the momentum mappings for these group actions are constant along any geodesic in  $\text{Imm}(M, \mathbb{R}^n)$ :

$$\begin{aligned}
 \forall X \in \mathfrak{X}(M) : \int_M \Phi(\text{Vol}(f), \text{Tr}(L^f)) \bar{g}(T f \cdot X, f_t) \text{vol}(g) & \quad \text{reparam. momenta} \\
 \text{or } \Phi(\text{Vol}(f), \text{Tr}(L^f)) g(f_t^\top) \text{vol}(g) \in \Gamma(T^*M \otimes_M \text{vol}(M)) & \quad \text{reparam. momentum} \\
 \int_M \Phi(\text{Vol}(f), \text{Tr}(L^f)) f_t \text{vol}(g) & \quad \text{linear momentum} \\
 \forall X \in \mathfrak{so}(n) : \int_M \Phi(\text{Vol}(f), \text{Tr}(L^f)) \bar{g}(X \cdot f, f_t) \text{vol}(g) & \quad \text{angular momenta} \\
 \text{or } \int_M \Phi(\text{Vol}(f), \text{Tr}(L^f)) (f \wedge f_t) \text{vol}(g) \in \bigwedge^2 \mathbb{R}^n \cong \mathfrak{so}(n)^* & \quad \text{angular momentum}
 \end{aligned}$$

## 6. THE GEODESIC EQUATION ON $B_i(M, \mathbb{R}^n)$

**6.1. The horizontal bundle and the metric on the quotient space.** Since  $\text{vol}(f^* \bar{g})$  and  $\text{Tr}(L^f)$  react equivariantly to the action of the group  $\text{Diff}(M)$ , every  $G^\Phi$ -metric is  $\text{Diff}(M)$ -invariant. As described in Section 1.3 it induces a Riemannian metric on  $B_i$  (off the singularities) such that the projection  $\pi : \text{Imm} \rightarrow B_i$  is a Riemannian submersion.

By definition, a tangent vector  $h$  to  $f \in \text{Imm}(M, \mathbb{R}^n)$  is horizontal if and only if it is  $G^\Phi$ -perpendicular to the  $\text{Diff}(M)$ -orbits. This is the case if and only if  $\bar{g}(h(x), T_x f \cdot X_x) = 0$  at every point  $x \in M$ . Therefore the horizontal bundle at the point  $f$  equals the set of sections of the normal bundle (see Section 2.9) along  $f$  and thus the metric on the horizontal bundle is given by

$$G_f^\Phi(h^{\text{hor}}, k^{\text{hor}}) = G_f^\Phi(a \cdot \nu, b \cdot \nu) = \int_M \Phi(\text{Vol}(f), \text{Tr}(L^f)(x)) a \cdot b \text{vol}(g).$$

The following lemma shows that every path in  $B_i$  corresponds to exactly one horizontal path in  $\text{Imm}$  and therefore the calculation of the geodesic equation can be done on the horizontal bundle instead of on  $B_i$ .

**Lemma.** *For any smooth path  $f$  in  $\text{Imm}$  there exists a smooth path  $\varphi$  in  $\text{Diff}(M)$  with  $\varphi(t, \cdot) = \text{Id}_M$  depending smoothly on  $f$  such that the path  $f(t, \varphi(t, x))$  is horizontal, i.e.  $\partial_t f(t, \varphi(t, x))$  lies in the horizontal bundle.*

The basic idea is to write the path  $\varphi$  as the integral curve of a time dependent vector field. This method is called the Moser-Trick. (see [15, Section 2.5])

**6.2. The geodesic equation on  $B_i(M, \mathbb{R}^n)$ .** As described in section 1.3, geodesics in  $B_i$  correspond to horizontal geodesics in  $\text{Imm}$ . A horizontal geodesic  $f$  in  $\text{Imm}$  has  $f_t = a \cdot \nu^f$  with  $a \in C^\infty(\mathbb{R} \times M)$ . The geodesic equation is given by

$$f_{tt} = \underbrace{a_t \cdot \nu}_{\text{normal}} + \underbrace{a \cdot \nu_t}_{\text{tang.}} = \frac{1}{2} H(a \cdot \nu, a \cdot \nu) - K(a \cdot \nu, a \cdot \nu),$$

see section 4.4. This equation splits into a normal and a tangential part. From the conservation of the reparametrization momentum, see section 4.5 and the previous section, it follows that the tangential part of the geodesic equation is satisfied automatically. We will nevertheless check this by hand. From section 5.1, where we calculated the metric gradients on  $\text{Imm}$ , we get

$$\begin{aligned} K_f(a \cdot \nu, a \cdot \nu) &= \left[ -\frac{\partial_1 \Phi}{\Phi} \int_M \text{Tr}(L) \cdot a \, \text{vol}(g) \right. \\ &\quad \left. + \frac{\partial_2 \Phi}{\Phi} (-\Delta a + a \, \text{Tr}(L^2)) - \text{Tr}(L) \cdot a \right] a \cdot \nu \\ H_f(a \cdot \nu, a \cdot \nu) &= \frac{1}{\Phi} T f \cdot \left[ (\partial_2 \Phi) a^2 \, \text{grad}^g(\text{Tr}(L)) - \text{grad}^g(\Phi a^2) \right] \\ &\quad + \left[ -\frac{1}{\Phi} \text{Tr}(L) \int_M \partial_1 \Phi a^2 \, \text{vol}(g) - \frac{1}{\Phi} \Delta((\partial_2 \Phi) a^2) \right. \\ &\quad \left. + \frac{1}{\Phi} (\partial_2 \Phi) \text{Tr}(L^2) a^2 - a^2 \text{Tr}(L) \right] \nu. \end{aligned}$$

From this we can easily read off the tangential part of the geodesic equation

$$a \cdot \nu_t = \frac{1}{2\Phi} T f \cdot \left[ (\partial_2 \Phi) a^2 \, \text{grad}^g(\text{Tr}(L)) - \text{grad}^g(\Phi a^2) \right].$$

To see that this equation is satisfied, we have to rewrite the right hand side using the following Leibnitz rule for the gradient,

$$g(\text{grad}^g(f_1 \cdot f_2), X) = g(f_1 \, \text{grad}^g f_2 + f_2 \, \text{grad}^g f_1, X).$$

This yields

$$\begin{aligned} a \cdot \nu_t &= \frac{1}{2\Phi} T f \cdot \left[ (\partial_2 \Phi) a^2 \, \text{grad}^g(\text{Tr}(L)) - \text{grad}^g(\Phi a^2) \right] \\ &= \frac{1}{2\Phi} T f \cdot \left[ (\partial_2 \Phi) a^2 \, \text{grad}^g(\text{Tr}(L)) - \Phi \cdot \text{grad}^g(a^2) - a^2 \cdot \text{grad}^g(\Phi) \right] \\ &= \frac{1}{2\Phi} T f \cdot \left[ (\partial_2 \Phi) a^2 \, \text{grad}^g(\text{Tr}(L)) - \Phi \cdot \text{grad}^g(a^2) - (\partial_2 \Phi) a^2 \, \text{grad}^g(\text{Tr}(L)) \right] \\ &= -\frac{1}{2\Phi} \Phi T f \cdot \text{grad}^g(a^2) = -T f \cdot a \cdot \text{grad}^g(a). \end{aligned}$$

By the variational formula for  $\nu$  in section 3.11 this equation is satisfied automatically. The normal part is given by

$$\begin{aligned} a_t &= \bar{g}\left(\frac{1}{2}H(a.\nu, a.\nu) - K(a.\nu, a.\nu), \nu\right) \\ &= \frac{1}{\Phi}\left[\frac{1}{2}\Phi a^2 \operatorname{Tr}(L^f) - \frac{1}{2}\operatorname{Tr}(L^f) \int_M (\partial_1 \Phi) a^2 \operatorname{vol}(f^* \bar{g}) - \frac{1}{2}\Delta((\partial_2 \Phi).a^2) \right. \\ &\quad \left. + (\partial_1 \Phi)a \int_M \operatorname{Tr}(L^f).a \operatorname{vol}(f^* \bar{g}) - \frac{1}{2}(\partial_2 \Phi) \operatorname{Tr}((L^f)^2)a^2 + (\partial_2 \Phi)a\Delta a\right]. \end{aligned}$$

We can rewrite this equation by expanding Laplacians of products as follows:

$$\Delta(a_1 a_2) = (\Delta a_1) a_2 - 2 \operatorname{Tr}^g(da_1 \otimes da_2) + a_1(\Delta a_2).$$

**Theorem.** *The geodesic equation of the almost local metric  $G^\Phi$  on  $B_i$  reads as*

$$\begin{aligned} f_t &= a.\nu^f, \\ a_t &= \frac{1}{\Phi}\left[\frac{1}{2}\Phi a^2 \operatorname{Tr}(L^f) - \frac{1}{2}\operatorname{Tr}(L^f) \int_M (\partial_1 \Phi) a^2 \operatorname{vol}(f^* \bar{g}) - \frac{1}{2}a^2 \Delta(\partial_2 \Phi) \right. \\ &\quad \left. + 2a \operatorname{Tr}^g(d(\partial_2 \Phi) \otimes da) + (\partial_2 \Phi) \operatorname{Tr}^g(da \otimes da) \right. \\ &\quad \left. + (\partial_1 \Phi)a \int_M \operatorname{Tr}(L^f).a \operatorname{vol}(f^* \bar{g}) - \frac{1}{2}(\partial_2 \Phi) \operatorname{Tr}((L^f)^2)a^2\right]. \end{aligned}$$

For the case of curves immersed in  $\mathbb{R}^2$ , this formula specializes to the formula given in [17, section 3.4]. (When verifying this, remember that  $\Delta = -D_s^2$  in the notation of [17].)

## 7. SECTIONAL CURVATURE ON SHAPE SPACE

To compute the sectional curvature we will use the following formula, which is valid in a chart:

$$\begin{aligned} R_0(a_1, a_2, a_1, a_2) &= G_0^\Phi(R_0(a_1, a_2)a_1, a_2) = \\ &= \frac{1}{2}d^2 G_0^\Phi(a_1, a_1)(a_2, a_2) - d^2 G_0^\Phi(a_1, a_2)(a_1, a_2) + \frac{1}{2}d^2 G_0^\Phi(a_2, a_2)(a_1, a_1) \\ &\quad + G_0^\Phi(\Gamma_0(a_1, a_1), \Gamma_0(a_2, a_2)) - G_0^\Phi(\Gamma_0(a_1, a_2), \Gamma_0(a_1, a_2)). \end{aligned}$$

Sectional curvature is given by

$$R_0(a_1, a_2, a_2, a_1) = -R_0(a_1, a_2, a_1, a_2).$$

Therefore we have to calculate the metric in a chart and calculate its second derivative.

**7.1. The almost local metric  $G^\Phi$  in a chart.** In the following section we will follow the method of [15]. First we will construct a local chart for  $B_i$ . Let  $f_0 : M \rightarrow \mathbb{R}^n$  be a fixed immersion, which will be the center of our chart. Consider the mapping

$$\begin{aligned} \psi &= \psi_{f_0} : C^\infty(M, (-\epsilon, \epsilon)) \rightarrow \operatorname{Imm}(M, \mathbb{R}^n) \\ \psi(a)(x) &= \exp_{f_0(x)}^{\bar{g}}(a(x).\nu^{f_0}(x)) = f_0(x) + a(x).\nu^{f_0}(x), \end{aligned}$$

where  $\epsilon$  is so small that  $\psi(a)$  is an immersion for each  $a$ .

Denote by  $\pi$  the projection from  $\text{Imm}(M, \mathbb{R}^n)$  to  $B_i(M, \mathbb{R}^n)$ . The inverse on its image of  $\pi \circ \psi : C^\infty(M, (-\epsilon, \epsilon)) \rightarrow B_i(M, \mathbb{R}^n)$  is then a smooth chart on  $B_i(M, \mathbb{R}^n)$ . We want to calculate the induced metric in this chart, i.e.

$$((\pi \circ \psi)^* G^\Phi)_a(b_1, b_2)$$

for any  $a \in C^\infty(M, (-\epsilon, \epsilon))$  and  $b_1, b_2 \in C^\infty(M)$ . We shall fix the function  $a$  and work with the ray of points  $t.a$  in this chart. Everything will revolve around the map:

$$f(t, x) = \psi(t.a)(x) = f_0(x) + t.a(x).\nu^{f_0}(x)$$

We shall use a fixed chart  $(u, U)$  on  $M$  with  $\partial_i = \frac{\partial}{\partial u^i}$ . Then in this chart, the pullback metric is given by

$$g|_U = \sum_{i,j}^{n-1} g_{ij} du^i \otimes du^j = \sum_{i,j}^{n-1} \bar{g}(\partial_i f, \partial_j f) du^i \otimes du^j,$$

the volume density by

$$\text{vol}(g) = \sqrt{\det(\bar{g}(\partial_i f, \partial_j f))} |du^1 \wedge \dots \wedge du^{n-1}|,$$

the second fundamental form by

$$s_{ij} = s(\partial_i, \partial_j) = \bar{g}(\nabla_{\partial_i} T f, \partial_j) = \bar{g}\left(\frac{\partial^2 f}{\partial_i \partial_j}, \nu^{f_0}\right),$$

and the mean curvature by  $\text{Tr}(L) = \sum_{i,j} g^{ij} s_{ij}$ . To calculate the metric  $G^\Phi$  in this chart we have to understand how

$$T_{t.a}\psi.b_1 = b_1(x).\nu^{f_0}(x)$$

splits into a tangential and horizontal part with respect to the immersion  $f(t, \cdot)$ . The tangential part locally has the form

$$T f.(T_{(t.a)}\psi.(b_1))^\top = \sum_{i=1}^{n-1} c^i \partial_i f(t, x),$$

where the coefficients  $c^i$  are given by

$$c^i = \sum_{j=1}^{n-1} g^{ij} \bar{g}(b_1(x).\nu^{f_0}(x), \partial_j f(t, x)).$$

Thus the horizontal part is

$$(T_{t.a}\psi.b_1).\nu = (T_{t.a}\psi.b_1) - T f.(T_{(t.a)}\psi.(b_1))^\top = b_1(x).\nu^{f_0}(x) - \sum_{i=1}^{n-1} c^i \partial_i f(t, x).$$

**Lemma.** *The expression of  $G^\Phi$  in the chart  $(\pi \circ \psi)^{-1}$  is:*

$$\begin{aligned} ((\pi \circ \psi_{f_0})^* G^\Phi)_{(t.a)}(b_1, b_2) &= G_{\pi(\psi(t.a))}^\Phi(T_{t.a}(\pi \circ \psi).b_1, T_{t.a}(\pi \circ \psi).b_2) \\ &= G_{\psi(t.a)}^\Phi((T_{t.a}\psi.b_1)^\perp.\nu, (T_{t.a}\psi.b_2)^\perp.\nu) \\ &= \int_M \Phi \bar{g}((T_{t.a}\psi.b_1)^\perp.\nu, (T_{t.a}\psi.b_2)^\perp.\nu) \text{vol}(g) \\ &= \int_M \Phi \left( b_1.b_2 - \sum_{i=1}^{n-1} c^i \bar{g}(\partial_i f(t, x), b_2(x).\nu^{f_0}(x)) \right) \text{vol}(g) \end{aligned}$$

**7.2. Second derivative of the  $G^\Phi$ -metric in the chart.** We will calculate

$$\partial_t^2|_0((\pi \circ \psi_{f_0})^* G^\Phi)_{(t,a)}(b_1, b_2).$$

We will use the following arguments repeatedly:

$$\partial_t|_0 \partial_j f = \partial_j \partial_t|_0 f = \partial_j(a \cdot \nu^{f_0}) = (\partial_j a) \nu^{f_0} + \underbrace{a(\partial_j \nu^{f_0})}_{\text{tang.}},$$

$$\bar{g}(b_1(x) \nu^{f_0}(x), \partial_j f(t, x))|_{t=0} = 0,$$

and consequently  $c_i|_{t=0} = 0$ .

$$\begin{aligned} \partial_t|_0 c_i &= \sum_{j=1}^{n-1} \partial_t|_0(g^{ij}).0 + \sum_{j=1}^{n-1} g^{ij} \partial_t|_0 \bar{g}(b_1 \nu^{f_0}, \partial_j f) \\ &= \sum_{j=1}^{n-1} g^{ij} \bar{g}(b_1 \nu^{f_0}, \partial_t|_0 \partial_j f) = \sum_{j=1}^{n-1} g^{ij} \bar{g}(b_1 \nu^{f_0}, \partial_j(a \cdot \nu^{f_0})) = \sum_{j=1}^{n-1} g^{ij} b_1 \partial_j a. \end{aligned}$$

Therefore

$$\begin{aligned} (b_1 \cdot b_2 - \sum_{i=1}^{n-1} c^i \bar{g}(\partial_i f, b_2 \cdot \nu^{f_0}))|_{t=0} &= b_1 \cdot b_2 \\ \partial_t|_0(b_1 \cdot b_2 - \sum_{i=1}^{n-1} c^i \bar{g}(\partial_i f, b_2 \cdot \nu^{f_0})) &= \\ &= - \sum_{i=1}^{n-1} (\partial_t|_0 c^i).0 - \sum_{i=1}^{n-1} 0 \cdot \partial_t|_0 \bar{g}(\partial_i f, b_2 \cdot \nu^{f_0}) = 0 \\ \partial_t^2|_0(b_1 \cdot b_2 - \sum_{i=1}^{n-1} c^i \bar{g}(\partial_i f, b_2 \cdot \nu^{f_0})) &= \\ &= - \sum_{i=1}^{n-1} (\partial_t^2|_0 c^i).0 - 2 \sum_{i=1}^{n-1} (\partial_t|_0 c^i) \partial_t|_0 \bar{g}(\partial_i f, b_2 \cdot \nu^{f_0}) - \sum_{i=1}^{n-1} 0 \cdot \partial_t^2|_0 \bar{g}(\partial_i f, b_2 \cdot \nu^{f_0}) \\ &= -2 \sum_{i=1}^{n-1} (\partial_t|_0 c^i) \bar{g}(\partial_i(a \cdot \nu^{f_0}), b_2 \cdot \nu^{f_0}) = -2 \sum_{i=1}^{n-1} (\partial_t|_0 c^i) (\partial_i a) b_2 \\ &= -2 b_1 b_2 \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} g^{ij} \partial_j a \cdot \partial_i a = -2 b_1 b_2 \|da\|_{g^{-1}}^2. \end{aligned}$$

The derivatives of  $\Phi$  are

$$\begin{aligned} \partial_t|_0(\Phi \circ (\text{Vol}, \text{Tr}(L))) &= (\partial_1 \Phi) \cdot (\partial_t|_0 \text{Vol}) + (\partial_2 \Phi) \cdot (\partial_t|_0 \text{Tr}(L)) \\ \partial_t^2|_0(\Phi \circ (\text{Vol}, \text{Tr}(L))) &= (\partial_1 \partial_1 \Phi) \cdot (\partial_t|_0 \text{Vol})^2 + (\partial_2 \partial_2 \Phi) \cdot (\partial_t|_0 \text{Tr}(L))^2 \\ &\quad + 2(\partial_1 \partial_2 \Phi) \cdot (\partial_t|_0 \text{Vol}) \cdot (\partial_t|_0 \text{Tr}(L)) + (\partial_1 \Phi) (\partial_t^2|_0 \text{Vol}) + (\partial_2 \Phi) (\partial_t^2|_0 \text{Tr}(L)). \end{aligned}$$

**Lemma.** *The second derivative of the  $G^\Phi$ -metric in the chart  $(\pi \circ \psi)^{-1}$  is given by:*

$$\begin{aligned} (1) \quad \partial_t^2|_0((\pi \circ \psi_{f_0})^* G^\Phi)_{(t,a)}(b_1, b_2) &= \left( d^2((\pi \circ \psi_{f_0})^* G^\Phi)(0)(a, a) \right)(b_1, b_2) \\ &= \int_M \dots b_1 \cdot b_2 \text{vol}(g) \end{aligned}$$

over the following expression

$$\begin{aligned} \dots = & \Phi \left( \frac{\partial_t^2|_0 \text{vol}}{\text{vol}} - 2 \|da\|_{g^{-1}}^2 \right) + (\partial_1 \Phi) \cdot \left( (\partial_t^2|_0 \text{Vol}) + 2(\partial_t|_0 \text{Vol}) \frac{\partial_t|_0 \text{vol}}{\text{vol}} \right) \\ & + (\partial_2 \Phi) \cdot \left( (\partial_t^2|_0 \text{Tr}(L)) + 2(\partial_t|_0 \text{Tr}(L)) \frac{\partial_t|_0 \text{vol}}{\text{vol}} \right) + (\partial_1 \partial_1 \Phi) \cdot (\partial_t|_0 \text{Vol})^2 \\ & + 2(\partial_1 \partial_2 \Phi) \cdot (\partial_t|_0 \text{Vol})(\partial_t|_0 \text{Tr}(L)) + (\partial_2 \partial_2 \Phi) \cdot (\partial_t|_0 \text{Tr}(L))^2. \end{aligned}$$

**7.3. Sectional curvature on shape space.** To understand the structure in the formulas for the sectional curvature tensor, we will use some facts from linear algebra.

**7.3.1. Lemma.** *Let  $V = C^\infty(M)$ , and let  $P$  and  $Q$  be bilinear and symmetric maps  $V \times V \rightarrow V$ . Then*

$$\begin{aligned} \boxplus(P, Q)(a_1 \wedge a_2, b_1 \wedge b_2) := & \frac{1}{2} (P(a_1, b_1)Q(a_2, b_2) - P(a_1, b_2)Q(a_2, b_1) \\ & + P(a_2, b_2)Q(a_1, b_1) - P(a_2, b_1)Q(a_1, b_2)) \end{aligned}$$

*defines a symmetric, bilinear map  $(V \wedge V) \otimes (V \wedge V) \rightarrow V$ .*

Also  $\boxplus(P, Q) = \boxplus(Q, P)$ . The symbol  $\boxplus$  stands for the Young tableau encoding the symmetries, see [7]. We have

$$\begin{aligned} \boxplus(P, Q)(a_1 \wedge a_2, a_1 \wedge a_2) \\ = & \frac{1}{2} P(a_1, a_1)Q(a_2, a_2) - P(a_1, a_2)Q(a_2, a_1) + \frac{1}{2} P(a_2, a_2)Q(a_1, a_1). \end{aligned}$$

$P$  is called positive semidefinite if for all  $x \in M$  and  $a \in C^\infty(M)$ ,  $P(a, a)(x) \geq 0$ .  $P$  is called negative semidefinite if  $-P$  is positive semidefinite. We will write  $P \geq 0, P \leq 0, P \lesssim 0$  if  $P$  is positive semidefinite, negative semidefinite or indefinite.

**7.3.2. Lemma.** *If  $P$  and  $Q$  are positive semidefinite bilinear and symmetric maps  $V \times V \rightarrow V$ , then also  $\boxplus(P, Q)$  is a positive semidefinite symmetric, bilinear map.*

*Proof.* To shorten notation, we will write for instance  $P_{12}$  instead of  $P(a_1, a_2)$ . The Cauchy inequality applied to  $P$  and  $Q$  gives us

$$P_{12}Q_{12} \leq \sqrt{P_{11}P_{22}Q_{11}Q_{22}},$$

and therefore we have

$$\begin{aligned} \boxplus(P, Q)(a_1 \wedge a_2, a_1 \wedge a_2) &= \frac{1}{2} P_{11}Q_{11} - P_{12}Q_{12} + \frac{1}{2} P_{22}Q_{22} \\ &\geq \frac{1}{2} P_{11}Q_{22} - \sqrt{P_{11}P_{22}Q_{11}Q_{22}} + \frac{1}{2} P_{22}Q_{11} \\ &= \frac{1}{2} \left( \sqrt{P_{11}Q_{22}} - \sqrt{P_{22}Q_{11}} \right)^2 \geq 0. \quad \square \end{aligned}$$

Let  $\lambda, \mu : V \rightarrow V$ . Then the map  $\lambda \otimes \mu : V \otimes V \rightarrow V$  is given by

$$(\lambda \otimes \mu)(a \otimes b) = \lambda(a) \cdot \mu(b),$$

where the multiplication is in  $V = C^\infty(M)$ . Denote by  $\lambda \vee \mu$  the symmetrization of the tensor product given by

$$\lambda \vee \mu = \frac{1}{2}(\lambda \otimes \mu + \mu \otimes \lambda).$$

We will make use of the following simplifications:

**7.3.3. Lemma.** *Let  $\lambda, \beta, \mu, \nu : V \rightarrow V$ . Then the bilinear symmetric map*

$$\boxplus(\lambda \vee \beta, \mu \vee \nu)$$

*satisfies the following properties:*

$$(S1) \quad \boxplus(\lambda \vee \mu, \lambda \vee \nu)(a_1 \wedge a_2, a_1 \wedge a_2) = -\frac{1}{4}(\lambda \otimes \mu)(a_1 \wedge a_2).(\lambda \otimes \nu)(a_1 \wedge a_2),$$

$$(S2) \quad \boxplus(\lambda \vee \mu, \lambda \otimes \lambda) = 0,$$

$$(S3) \quad \boxplus(\lambda \otimes \lambda, \mu \vee \nu)(a_1 \wedge a_2, a_1 \wedge a_2) = \frac{1}{2}(\lambda \otimes \mu)(a_1 \wedge a_2).(\lambda \otimes \nu)(a_1 \wedge a_2).$$

*Proof.* For the proof of simplification (S1) we calculate:

$$\begin{aligned} & \boxplus(\lambda \vee \mu, \lambda \vee \nu)(a_1 \wedge a_2, a_1 \wedge a_2) \\ &= \frac{1}{2}(\lambda \otimes \mu \otimes \lambda \otimes \nu) \left[ a_1 \otimes a_1 \otimes a_2 \otimes a_2 + a_2 \otimes a_2 \otimes a_1 \otimes a_1 \right. \\ & \quad - \frac{1}{2}a_1 \otimes a_2 \otimes a_1 \otimes a_2 - \frac{1}{2}a_1 \otimes a_2 \otimes a_2 \otimes a_1 \\ & \quad \left. - \frac{1}{2}a_2 \otimes a_1 \otimes a_1 \otimes a_2 - \frac{1}{2}a_2 \otimes a_1 \otimes a_2 \otimes a_1 \right] \end{aligned}$$

Using the symmetries of the quasilinear mapping  $\lambda \otimes \mu \otimes \lambda \otimes \mu$ , we can swap the first and third position in the tensor product of the two summands in the first line. Then the expression inside the square brackets equals  $-\frac{1}{2}(a_1 \wedge a_2) \otimes (a_1 \wedge a_2)$ .

Since  $\lambda \otimes \lambda$  vanishes when applied to elements of  $V \wedge V$ , simplification (S2) is a direct consequence of (S1).

For the proof of simplification (S3) we calculate:

$$\begin{aligned} & \boxplus(\lambda \otimes \lambda, \mu \vee \nu)(a_1 \wedge a_2, a_1 \wedge a_2) \\ &= \frac{1}{2}(\lambda \otimes \lambda \otimes \mu \otimes \nu) \left[ a_1 \otimes a_1 \otimes a_2 \otimes a_2 + a_2 \otimes a_2 \otimes a_1 \otimes a_1 \right. \\ & \quad \left. - a_1 \otimes a_2 \otimes a_1 \otimes a_2 - a_1 \otimes a_2 \otimes a_2 \otimes a_1 \right] \end{aligned}$$

Using symmetries as above, we can replace third summand  $a_1 \otimes a_2 \otimes a_1 \otimes a_2$  by  $a_2 \otimes a_1 \otimes a_2 \otimes a_1$ , because the first two tensor components of  $\lambda \otimes \lambda \otimes \mu \otimes \nu$  are equal. Then, swapping the second and third position in all tensor products, we get

$$\begin{aligned} & \boxplus(\lambda \otimes \lambda, \mu \otimes \nu)(a_1 \wedge a_2, a_1 \wedge a_2) \\ &= \frac{1}{2}(\lambda \otimes \mu \otimes \lambda \otimes \nu) \left[ a_1 \otimes a_2 \otimes a_1 \otimes a_2 + a_2 \otimes a_1 \otimes a_2 \otimes a_1 \right. \\ & \quad \left. - a_2 \otimes a_1 \otimes a_1 \otimes a_2 - a_1 \otimes a_2 \otimes a_2 \otimes a_1 \right] \end{aligned}$$

The expression inside the square brackets equals  $(a_1 \wedge a_2) \otimes (a_1 \wedge a_2)$ .  $\square$

For orthonormal  $a_1, a_2$  sectional curvature is the negative of the curvature tensor  $R_0(a_1, a_2, a_1, a_2)$ . We will use the following *formula for the curvature tensor*, which

is valid in a chart:

$$(1) \quad \begin{aligned} R_0(a_1, a_2, a_1, a_2) &= G_0^\Phi(R_0(a_1, a_2)a_1, a_2) = \\ &\frac{1}{2}d^2G_0^\Phi(a_1, a_1)(a_2, a_2) - d^2G_0^\Phi(a_1, a_2)(a_1, a_2) + \frac{1}{2}d^2G_0^\Phi(a_2, a_2)(a_1, a_1) \\ &+ G_0^\Phi(\Gamma_0(a_1, a_1), \Gamma_0(a_2, a_2)) - G_0^\Phi(\Gamma_0(a_1, a_2), \Gamma_0(a_1, a_2)). \end{aligned}$$

Looking at Formula (1) from section 7.2 we can express the second derivative of the metric  $G^\Phi$  in the chart as

$$\begin{aligned} &\left(d^2(\pi \circ \psi_{f_0})^*G^\Phi(0)(a_1, a_2)\right)(b_1, b_2) \\ &= \int_M \left( \Phi.P_1(a_1, a_2) + (\partial_1\Phi)P_2(a_1, a_2) + (\partial_2\Phi)P_3(a_1, a_2) + (\partial_1\partial_1\Phi)P_4(a_1, a_2) \right. \\ &\quad \left. + (\partial_1\partial_2\Phi)P_5(a_1, a_2) + (\partial_2\partial_2\Phi)P_6(a_1, a_2) \right) P(b_1, b_2) \text{vol}(g), \end{aligned}$$

where  $P(b_1, b_2) = b_1.b_2$ , so  $P = \text{id} \otimes \text{id}$ , and where the  $P_i$  are obtained by symmetrizing the terms in Formula (1) from section 7.2.

For the rest of this section, we do not note the pullback via the chart anymore, writing  $G_0^\Phi$  instead of  $((\pi \circ \psi_{f_0})^*G^\Phi)(0)$ , for example. To further shorten our notation, we write  $L$  instead of  $L^{f_0}$  and  $g$  instead of  $g_0$ . The following terms are calculated using the variational formulas from section 3.

$$\begin{aligned} P_1(a, a) &= \frac{\partial_t^2|_0 \text{vol}}{\text{vol}} - 2\|da\|_{g^{-1}}^2 = a^2(\text{Tr}(L)^2 - \text{Tr}(L^2)) - \|da\|_{g^{-1}}^2 \\ P_2(a, a) &= (\partial_t^2|_0 \text{Vol}) + 2(\partial_t|_0 \text{Vol}) \frac{\partial_t|_0 \text{vol}}{\text{vol}} \\ &= \int_M a^2(\text{Tr}(L)^2 - \text{Tr}(L^2)) + \int_M \|da\|_{g^{-1}}^2 \text{vol}(g) \\ &\quad + 2\text{Tr}(L).a \int_M \text{Tr}(L).a \text{vol}(g) \\ P_3(a, a) &= (\partial_t^2|_0 \text{Tr}(L)) + 2(\partial_t|_0 \text{Tr}(L)) \frac{\partial_t|_0 \text{vol}}{\text{vol}} \\ &= 2a^2 \text{Tr}(L^3) + 4a \text{Tr}(L.g^{-1}.\nabla^2 a) + 2\text{Tr}(g^{-1}(da \otimes da)L) \\ &\quad - \|da\|_{g^{-1}}^2 \text{Tr}(L) + 2a \text{Tr}^g(\nabla_{\text{grad } a} s) \\ &\quad + 2(-\Delta a + a \text{Tr}(L^2))(-\text{Tr}(L).a) \\ &= 2a^2 \text{Tr}(L^3) + 4a \text{Tr}(L.g^{-1}.\nabla^2 a) + 2\text{Tr}(g^{-1}(da \otimes da)L) \\ &\quad - \|da\|_{g^{-1}}^2 \text{Tr}(L) + 2a \text{Tr}^g(\nabla_{\text{grad } a} s) \\ &\quad + 2\text{Tr}(L)a\Delta a - 2\text{Tr}(L) \text{Tr}(L^2).a^2 \\ P_4(a, a) &= (\partial_t|_0 \text{Vol})^2 = \left( \int_M \text{Tr}(L).a \text{vol}(g) \right)^2 \\ P_5(a, a) &= 2(\partial_t|_0 \text{Vol})(\partial_t|_0 \text{Tr}(L)) = 2 \int_M -\text{Tr}(L).a \text{vol}(g) (-\Delta a + a \text{Tr}(L^2)) \\ &= 2\Delta a \int_M \text{Tr}(L).a \text{vol}(g) - 2\text{Tr}(L^2)a \int_M \text{Tr}(L).a \text{vol}(g) \end{aligned}$$



$$\begin{aligned} P_6(a, a) &= (\partial_t|_0 \operatorname{Tr}(L))^2 = (-\Delta a + a \operatorname{Tr}(L^2))^2 \\ &= (\Delta a)^2 - 2a\Delta a \operatorname{Tr}(L^2) + a^2 \operatorname{Tr}(L^2)^2 \end{aligned}$$

Then the *first part of the curvature tensor* is given by

$$\begin{aligned} &\frac{1}{2}d^2G_0^\Phi(a_1, a_1)(a_2, a_2) - d^2G_0^\Phi(a_1, a_2)(a_1, a_2) + \frac{1}{2}d^2G_0^\Phi(a_2, a_2)(a_1, a_1) \\ &= \int_M (\Phi \cdot \boxplus(P_1, P) + (\partial_1 \Phi) \boxplus(P_2, P) + (\partial_2 \Phi) \boxplus(P_3, P) \\ &\quad + (\partial_1 \partial_1 \Phi) \boxplus(P_4, P) + (\partial_1 \partial_2 \Phi) \boxplus(P_5, P) + (\partial_2 \partial_2 \Phi) \boxplus(P_6, P)) \operatorname{vol}(g) \\ &\quad \cdot (a_1 \wedge a_2, a_1 \wedge a_2). \end{aligned}$$

Note that  $P$  is positive definite, so that  $\boxplus(P_i, P)$  is positive semidefinite if  $P_i$  is positive semidefinite. We can always assume that  $\Phi$  is positive because otherwise  $G^\Phi$  would not be a Riemannian metric.

$$\boxed{P_1 P} \quad P_1 = P_1^1 + P_1^2,$$

with

$$\begin{aligned} P_1^1 &= (\operatorname{Tr}(L)^2 - \operatorname{Tr}(L^2)) \operatorname{id} \otimes \operatorname{id} \\ P_1^2 &= -\operatorname{Tr}^g(d \otimes d) \end{aligned}$$

Applying simplification (S3) to  $\boxplus(P_1^1, P)$  and  $\boxplus(P_1^2, P)$ , we get

$$\boxplus(P_1^1, P) = \frac{1}{2}(\operatorname{Tr}(L)^2 - \operatorname{Tr}(L^2))(\operatorname{id} \otimes \operatorname{id})^2 = 0$$

on  $(V \wedge V) \otimes (V \wedge V)$  and

$$\begin{aligned} \boxplus(P_1^2, P) &= -\frac{1}{2} \operatorname{Tr}^g((\operatorname{id} \otimes d)^2), \\ \boxplus(P_1^2, P)(a_1 \wedge a_2, a_1 \wedge a_2) &= -\frac{1}{2} \|a_1 da_2 - a_2 da_1\|_{g^{-1}}^2 \leq 0. \end{aligned}$$

Therefore we have

$$\int_M \Phi \cdot \boxplus(P_1, P)(a_1 \wedge a_2, a_1 \wedge a_2) \operatorname{vol}(g) \leq 0.$$

$$\boxed{P_2 P} \quad P_2 = P_2^1 + P_2^2 + P_2^3$$

with

$$\begin{aligned} P_2^1 &= \int_M (\operatorname{id} \otimes \operatorname{id})(\operatorname{Tr}(L)^2 - \operatorname{Tr}(L^2)) \operatorname{vol}(g) \\ P_2^2 &= 2 \operatorname{Tr}(L)(\operatorname{id} \vee \int_M \operatorname{Tr}(L) \operatorname{id} \operatorname{vol}(g)) \\ P_2^3 &= \int_M \operatorname{Tr}^g(d \otimes d) \operatorname{vol}(g) \end{aligned}$$

$P_2^1$  is indefinite. Applying Simplification (S2) we get  $\boxplus(P_2^2, P) = 0$ .  $P_2^3$  and therefore also  $\boxplus(P_2^3, P)$  is positive semidefinite. Therefore

$$\begin{aligned} \int_M (\partial_1 \Phi) \boxplus(P_2^1, P)(a_1 \wedge a_2, a_1 \wedge a_2) \operatorname{vol}(g) &\lesssim 0, \\ \int_M (\partial_1 \Phi) \boxplus(P_2^3, P)(a_1 \wedge a_2, a_1 \wedge a_2) \operatorname{vol}(g) &\geq 0 \\ \boxed{P_3 P} \quad P_3 &= P_3^1 + P_3^2 + P_3^3, \end{aligned}$$

with

$$\begin{aligned} P_3^1 &= 2 \operatorname{id} \vee \left( \operatorname{Tr}(L^3) \operatorname{id} + 2 \operatorname{Tr}(Lg^{-1} \nabla^2(\operatorname{id})) + \operatorname{Tr}^g(\nabla_{\operatorname{grad} id} s) \right. \\ &\quad \left. + \operatorname{Tr}(L) \Delta(\operatorname{id}) - \operatorname{Tr}(L) \operatorname{Tr}(L)^2 \operatorname{id} \right) \\ P_3^2 &= 2 \operatorname{Tr}(g^{-1}(d \otimes d)L) \\ P_3^3 &= -\operatorname{Tr}^g(d \otimes d) \operatorname{Tr}(L) \end{aligned}$$

Applying Simplification (S2) we get that  $\boxplus(P_3^1, P)$  vanishes. Furthermore,

$$\begin{aligned} \boxplus(P_3^2, P)(a_1 \wedge a_2, a_1 \wedge a_2) &= a_1^2 \operatorname{Tr}(g^{-1}(da_2 \otimes da_2).L) \\ &\quad - 2a_1 a_2 \operatorname{Tr}(g^{-1}(da_1 \otimes da_2).L) \\ &\quad + a_2^2 \operatorname{Tr}(g^{-1}(da_1 \otimes da_1).L) \\ &= g_2^0((a_1 da_2 - a_2 da_1) \otimes (a_1 da_2 - a_2 da_1), s) \lesssim 0 \\ \boxplus(P_3^3, P)(a_1 \wedge a_2, a_1 \wedge a_2) &= -\frac{1}{2} \|a_1 da_2 - a_2 da_1\|_{g^{-1}}^2 \operatorname{Tr}(L) \lesssim 0 \\ \boxed{P_4 P} \quad P_4 &= \int_M \operatorname{Tr}(L) \operatorname{id} \operatorname{vol}(g) \otimes \int_M \operatorname{Tr}(L) \operatorname{id} \operatorname{vol}(g) \end{aligned}$$

Applying Simplification (S3) we get

$$\boxplus(P_4, P) = \frac{1}{2} \left( \operatorname{id} \otimes \int_M \operatorname{Tr}(L) \operatorname{id} \operatorname{vol}(g) \right)^2.$$

Therefore, if  $\partial_1 \partial_1 \Phi \geq 0$

$$\begin{aligned} \int_M \partial_1 \partial_1 \Phi \boxplus(P_4, P)(a_1 \wedge a_2, a_1 \wedge a_2) \operatorname{vol}(g) &\geq 0, \\ \boxed{P_5 P} \quad P_5 &= P_5^1 + P_5^2 \end{aligned}$$

with

$$\begin{aligned} P_5^1 &= 2 \left( \Delta \vee \int_M \operatorname{Tr}(L) \operatorname{id} \operatorname{vol}(g) \right) \\ P_5^2 &= -2 \operatorname{Tr}(L^2) \left( \operatorname{id} \vee \int_M \operatorname{Tr}(L) a_2 \operatorname{vol}(g) \right) \end{aligned}$$

Applying Simplification (S3) we get that  $\boxplus(P_5^1, P)$  is the indefinite form given by

$$\boxplus(P_5^1, P) = (\operatorname{id} \otimes \Delta) \otimes \left( \operatorname{id} \otimes \int_M \operatorname{Tr}(L) \operatorname{id} \operatorname{vol}(g) \right)$$

Simplification (S2) gives  $\boxplus(P_5^2, P) = 0$ . Therefore

$$\int_M (\partial_1 \partial_2 \Phi) \boxplus(P_5^1, P)(a_1 \wedge a_2, a_1 \wedge a_2) \text{vol}(g) \lesssim 0.$$

$P_6 P$

 $P_6 = P_6^1 + P_6^2$

with

$$\begin{aligned} P_6^1 &= \Delta \otimes \Delta \\ P_6^2 &= \text{Tr}(L^2)^2 \text{id} \otimes \text{id} \\ P_6^3 &= -2 \text{Tr}(L^2) \text{id} \vee \Delta \end{aligned}$$

Applying Simplification (S2) we get that  $\boxplus(P_6^2, P)$  and  $\boxplus(P_6^3, P)$  vanish. Simplification (S3) gives

$$\boxplus(P_6^1, P) = \frac{1}{2}(\text{id} \otimes \Delta)^2$$

We get

$$\int_M (\partial_2 \partial_2 \Phi) \boxplus(P_6, P)(a_1 \wedge a_2, a_1 \wedge a_2) \text{vol}(g) \geq 0$$

if  $\partial_2 \partial_2 \Phi \geq 0$ .

Now we come to the *second part of the curvature tensor*  $R_0(a_1, a_2, a_1, a_2)$ , which is given by

$$G_0(\Gamma_0(a_1, a_1), \Gamma_0(a_2, a_2)) - G_0(\Gamma_0(a_1, a_2), \Gamma_0(a_1, a_2)).$$

From the geodesic equation calculated in section 6, which is given by

$$\begin{aligned} a_t = \Gamma_0(a, a) &= \frac{1}{\Phi} \left[ \frac{1}{2} \Phi a^2 \text{Tr}(L) - \frac{1}{2} \text{Tr}(L) \int_M (\partial_1 \Phi) a^2 \text{vol}(g) - \frac{1}{2} a^2 \Delta(\partial_2 \Phi) \right. \\ &\quad + 2a \text{Tr}^g(d(\partial_2 \Phi) \otimes da) + (\partial_2 \Phi) \text{Tr}^g(da \otimes da) \\ &\quad \left. + (\partial_1 \Phi) a \int_M \text{Tr}(L).a \text{vol}(g) - \frac{1}{2} (\partial_2 \Phi) \text{Tr}(L^2) a^2 \right], \end{aligned}$$

we can extract the Christoffel symbol by symmetrization and get

$$\Gamma_0(a_1, a_2) = \frac{1}{\Phi} \sum_{i=1}^5 Q_i(a_1, a_2),$$

where  $Q_1, \dots, Q_5$  are the symmetrizations of the summands in the geodesic equation.  $Q_i$  are given by

$$\begin{aligned} Q_1 &= \frac{1}{2} \left( \Phi \text{Tr}(L) - \Delta(\partial_2 \Phi) - (\partial_2 \Phi) \text{Tr}(L^2) \right) \text{id} \otimes \text{id}, \\ Q_2 &= -\frac{1}{2} \text{Tr}(L) \int_M (\partial_1 \Phi) \text{id} \otimes \text{id} \text{vol}(g), \\ Q_3 &= 2 \text{id} \vee \text{Tr}^g(d(\partial_2 \Phi) \otimes d) \\ Q_4 &= (\partial_1 \Phi) \text{id} \vee \int_M \text{Tr}(L) \text{id} \text{vol}(g), \end{aligned}$$

$$Q_5 = (\partial_2 \Phi) \operatorname{Tr}^g(d \otimes d).$$

Then

$$\begin{aligned} & G_0(\Gamma_0(a_1, a_1), \Gamma_0(a_2, a_2)) - G_0(\Gamma_0(a_1, a_2), \Gamma_0(a_1, a_2)) \\ &= \int_M \frac{1}{\Phi} \sum_i \boxplus(Q_i, Q_i)(a_1 \wedge a_2, a_1 \wedge a_2) \operatorname{vol}(g) \\ &+ \int_M \frac{2}{\Phi} \sum_{i < j} \boxplus(Q_i, Q_j)(a_1 \wedge a_2, a_1 \wedge a_2) \operatorname{vol}(g). \end{aligned}$$

The contribution of the following terms to  $R_0(a_1, a_2, a_1, a_2)$  is  $\int_M \frac{1}{\Phi} \dots \operatorname{vol}(g)$  over the terms listed.

$$\boxed{Q_1 Q_1} \quad \boxplus(Q_1, Q_1) = 0$$

according to Simplification (S2).

$$\boxed{Q_2 Q_2} \quad \boxplus(Q_2, Q_2)(a_1 \wedge a_2, a_1 \wedge a_2) = \frac{\operatorname{Tr}(L)^2}{4} \left[ \int_M (\partial_1 \Phi) a_1^2 \operatorname{vol}(g) \cdot \int_M (\partial_1 \Phi) a_2^2 \operatorname{vol}(g) - \left( \int_M (\partial_1 \Phi) a_1 a_2 \operatorname{vol}(g) \right)^2 \right]$$

which is positive by the Cauchy-Schwarz inequality, assuming that  $\partial_1 \Phi \geq 0$ .

$$\begin{aligned} \boxed{Q_3 Q_3} \quad \boxplus(Q_3, Q_3)(a_1 \wedge a_2, a_1 \wedge a_2) &= \\ &= - \left( (\operatorname{id} \otimes \operatorname{Tr}^g(d(\partial_2 \Phi) \otimes d))(a_1 \wedge a_2) \right)^2 \\ &= -g^{-1}(d(\partial_2 \Phi), a_1 da_2 - a_2 da_1)^2 \leq 0 \end{aligned}$$

according to Simplification (S1).

$$\boxed{Q_4 Q_4} \quad \boxplus(Q_4, Q_4) = -\frac{1}{4}(\partial_1 \Phi)^2 (\operatorname{id} \otimes \int_M \operatorname{Tr}(L) \operatorname{id} \operatorname{vol}(g))^2 \leq 0$$

according to Simplification (S1).

$$\begin{aligned} \boxed{Q_5 Q_5} \quad \boxplus(Q_5, Q_5) &= (\partial_2 \Phi)^2 (\|da_1\|_{g^{-1}}^2 \|da_2\|_{g^{-1}}^2 - g^{-1}(da_1, da_2)^2) \\ &= (\partial_2 \Phi)^2 \|da_1 \wedge da_2\|_{g_2^0}^2 \geq 0 \end{aligned}$$

by the Cauchy-Schwarz inequality.

The contribution of the following terms to  $R_0(a_1, a_2, a_1, a_2)$  is  $\int_M \frac{2}{\Phi} \dots \operatorname{vol}(g)$  over the terms listed.

$$\begin{aligned} \boxed{Q_1 Q_2} \quad \boxplus(Q_1, Q_2) &= -\frac{1}{4}(\Phi \operatorname{Tr}(L)^2 - \operatorname{Tr}(L) \Delta(\partial_2 \Phi) - \operatorname{Tr}(L) \operatorname{Tr}(L^2)(\partial_2 \Phi)) \\ &\quad \cdot \boxplus(\operatorname{id} \otimes \operatorname{id}, \int_M (\partial_1 \Phi) \operatorname{id} \otimes \operatorname{id} \operatorname{vol}(g)), \end{aligned}$$

where the second factor is  $\geq 0$  assuming that  $\partial_1 \Phi \geq 0$ .

$$\boxed{Q_1 Q_3} \quad \boxplus(Q_1, Q_3) = 0$$

according to Simplification (S2).

$$\boxed{Q_1 Q_4} \quad \boxplus(Q_1, Q_4) = 0$$

according to Simplification (S2).

$$\boxed{Q_1 Q_5} \quad \boxplus(Q_1, Q_5) = \frac{1}{4} (\Phi \operatorname{Tr}(L)(\partial_2 \Phi) - (\partial_2 \Phi) \Delta(\partial_2 \Phi) - \operatorname{Tr}(L^2)(\partial_2 \Phi)^2) \cdot \|a_1 da_2 - a_2 da_1\|_{g^{-1}}^2$$

$$\boxed{Q_2 Q_3} \quad \boxplus(Q_2, Q_3) \lesssim 0$$

$$\boxed{Q_2 Q_4} \quad \boxplus(Q_2, Q_4) = -\frac{1}{2}(\partial_1 \Phi) \operatorname{Tr}(L) \cdot \boxplus \left( \int_M (\partial_1 \Phi) \operatorname{id} \otimes \operatorname{id} \operatorname{vol}(g), \operatorname{id} \vee \int_M \operatorname{Tr}(L) \operatorname{id} \operatorname{vol}(g) \right)$$

This form is indefinite, but we have

$$\int_M \frac{2}{\Phi} \boxplus(Q_2, Q_4) \operatorname{vol}(g) = -\boxplus(\tilde{Q}_2, \tilde{Q}_4),$$

with the positive semidefinite form

$$\tilde{Q}_2 = \int_M (\partial_1 \Phi) \operatorname{id} \otimes \operatorname{id} \operatorname{vol}(g),$$

and the form

$$\tilde{Q}_4 = \int_M \operatorname{Tr}(L) \frac{1}{\Phi} (\partial_1 \Phi) \operatorname{id} \operatorname{vol}(g) \vee \int_M \operatorname{Tr}(L) \operatorname{id} \operatorname{vol}(g),$$

which is positive semidefinite if  $\frac{\partial_1 \Phi}{\Phi}$  is a non-negative constant.

$$\boxed{Q_2 Q_5} \quad \boxplus(Q_2, Q_5) = -\frac{1}{2}(\partial_2 \Phi) \operatorname{Tr}(L) \cdot \boxplus \left( \int_M (\partial_1 \Phi) \operatorname{id} \otimes \operatorname{id} \operatorname{vol}(g), \operatorname{Tr}^g(d \otimes d) \right) \lesssim 0,$$

because of the factor  $(\partial_2 \Phi) \operatorname{Tr}(L)$ . But the factor

$$\boxplus \left( \int_M (\partial_1 \Phi) \operatorname{id} \otimes \operatorname{id} \operatorname{vol}(g), \operatorname{Tr}^g(d \otimes d) \right)$$

is positive definite.

$$\boxed{Q_3 Q_4} \quad \boxplus(Q_3, Q_4) \lesssim 0$$

$$\begin{aligned} \boxed{Q_3 Q_5} \quad \boxplus(Q_3, Q_5)(a_1 \wedge a_2, a_1 \wedge a_2) &= \\ &= (\partial_2 \Phi) \left( a_1 g^{-1}(d(\partial_2 \Phi), da_1) \|da_2\|_{g^{-1}}^2 - (a_1 g^{-1}(d(\partial_2 \Phi), da_2) + \right. \\ &\quad \left. a_2 g^{-1}(d(\partial_2 \Phi), da_1)) g^{-1}(da_1, da_2) + a_2 g^{-1}(d(\partial_2 \Phi), da_2) \|da_1\|_{g^{-1}}^2 \right) \end{aligned}$$

$$= (\partial_2 \Phi) g_2^0 (d(\partial_2 \Phi) \otimes (a_1 da_2 - a_2 da_1), da_1 \wedge da_2) \lesssim 0$$

$$\boxed{Q_4 Q_5} \quad \boxplus(Q_4, Q_5) \lesssim 0$$

We are now able to compile a list of all negative, positive and indefinite terms of  $R_0(a_1, a_2, a_1, a_2)$ . Remember that negative terms of  $R_0(a_1, a_2, a_1, a_2)$  make a positive contribution to sectional curvature. Positive sectional curvature is connected to the vanishing of geodesic distance because the space wraps up on itself in tighter and tighter ways.

$$\boxed{P_4 P} \quad \boxed{P_6 P} \quad \boxed{Q_2 Q_2} \quad \boxed{Q_5 Q_5} \text{ are positive, assuming } \partial_1 \Phi, \partial_1 \partial_1 \Phi, \partial_2 \partial_2 \Phi \geq 0.$$

$$\boxed{P_1 P} \quad \boxed{Q_3 Q_3} \quad \boxed{Q_4 Q_4} \quad \boxed{Q_1 Q_2} \text{ are the negative, assuming that } \partial_1 \Phi \geq 0.$$

$$\boxed{Q_2 Q_4} \text{ is negative assuming that } \frac{\partial_1 \Phi}{\Phi} \text{ is a non-negative constant, and indefinite otherwise.}$$

$$\boxed{Q_2 Q_5} \text{ is negative assuming that } \text{Tr}(L)(\partial_2 \Phi) \text{ is positive, and indefinite otherwise.}$$

$$\boxed{P_2 P} \quad \boxed{P_3 P} \quad \boxed{P_5 P} \quad \boxed{Q_1 Q_5} \quad \boxed{Q_2 Q_3} \quad \boxed{Q_3 Q_4} \quad \boxed{Q_3 Q_5} \quad \boxed{Q_4 Q_5} \text{ are indefinite.}$$

## 8. GEODESIC DISTANCE ON $B_i(M, \mathbb{R}^n)$

We will state some conditions on  $\Phi$  ensuring that the almost local metric  $G^\Phi$  induces non-vanishing geodesic distance on  $B_i$ . The proofs are based on a comparison between the  $G^\Phi$ -length of a path and its area swept out. In the last part we will use the vector space structure of  $\mathbb{R}^n$  to define a Fréchet metric on shape space  $B_i(M, \mathbb{R}^n)$ . In 8.7 it is shown how this metric is related to a  $L^\infty$  Finsler metric, and in 8.8 the Fréchet metric is compared to almost local metrics.

The main results are in section 8.6 and in section 8.8.

**8.1. Geodesic distance on  $B_i$ .** Geodesic distance on  $B_i$  is given by

$$\text{dist}_{B_i}^{G^\Phi}(f_0, f_1) = \inf_f L_{B_i}^{G^\Phi}(f),$$

where the infimum is taken over all  $f : [0, 1] \rightarrow B_i$  with  $f(0) = f_0$  and  $f(1) = f_1$ .  $L_{B_i}^{G^\Phi}$  is the length of paths in  $B_i$  given by

$$L_{B_i}^{G^\Phi}(f) = \int_0^1 \sqrt{G_f^\Phi(f_t, f_t)} dt \quad \text{for } f : [0, 1] \rightarrow B_i.$$

Letting  $\pi : \text{Imm} \rightarrow B_i$  denote the projection, we have

$$L_{B_i}^{G^\Phi}(\pi \circ f) = L_{\text{Imm}}^{G^\Phi}(f) = \int_0^1 \sqrt{G_f^\Phi(f_t, f_t)} dt \quad \text{for horizontal } f : [0, 1] \rightarrow \text{Imm}.$$

By non-vanishing geodesic distance on  $B_i$  we mean that  $\text{dist}_{B_i}^{G^\Phi}$  separates points.

**8.2. Area swept out.** For  $f : [0, 1] \rightarrow \text{Imm}$  we have

$$(\text{area swept out by } f) = \int_{[0,1] \times M} \text{vol}(f(\cdot, \cdot)^* \bar{g}).$$

If  $f$  is horizontal, then this integral can be rewritten as

$$(\text{area swept out by } f) = \int_0^1 \int_M \|f_t\| \text{vol}(f(t, \cdot)^* \bar{g}) dt =: \int_0^1 \int_M \|f_t\| \text{vol}(g) dt.$$

**8.3. Lemma** (First area swept out bound). *For an almost local metric  $G^\Phi$  satisfying*

$$\Phi(\text{Vol}, \text{Tr}(L)) \geq C_1 \quad \text{for } C_1 > 0.$$

*and a horizontal path  $f : [0, 1] \rightarrow \text{Imm}$ , we have the area swept out bound*

$$\sqrt{C_1} (\text{area swept out by } f) \leq \max_t \sqrt{\text{Vol}(f(t))} \cdot L_{\text{Imm}}^{G^\Phi}(f)$$

The proof is an adaptation of the one given in [15, section 3.4] for the  $G^A$ -metric.

*Proof.*

$$\begin{aligned} L_{\text{Imm}}^{G^\Phi}(f) &= \int_0^1 \sqrt{G_f^\Phi(f_t, f_t)} dt \\ &= \int_0^1 \left( \int_M \Phi \|f_t\|^2 \text{vol}(g) \right)^{\frac{1}{2}} dt \geq \sqrt{C_1} \int_0^1 \left( \int_M \|f_t\|^2 \text{vol}(g) \right)^{\frac{1}{2}} dt \\ &\geq \sqrt{C_1} \int_0^1 \left( \int_M \text{vol}(g) \right)^{-\frac{1}{2}} \int_M 1 \cdot \|f_t\| \text{vol}(g) dt \\ &\geq \sqrt{C_1} \min_t \left( \int_M \text{vol}(g) \right)^{-\frac{1}{2}} \int_0^1 \int_M 1 \cdot \|f_t\| \text{vol}(g) dt \\ &= \sqrt{C_1} \left( \max_t \int_M \text{vol}(g) \right)^{-\frac{1}{2}} \int_0^1 \int_M 1 \cdot \|f_t\| \text{vol}(g) dt \quad \square \end{aligned}$$

**8.4. Lemma** (Lipschitz continuity of  $\sqrt{\text{Vol}}$ ). *For an almost local metric  $G^\Phi$ , the condition*

$$\Phi(\text{Vol}, \text{Tr}(L)) \geq C_2 \text{Tr}(L)^2$$

*implies the Lipschitz continuity of the map*

$$\sqrt{\text{Vol}} : (B_i, \text{dist}_{G^\Phi}^{B_i}) \rightarrow \mathbb{R}_{\geq 0}$$

*by the inequality holding for  $f_1$  and  $f_2$  in  $B_i$ :*

$$\sqrt{\text{Vol}(f_1)} - \sqrt{\text{Vol}(f_2)} \leq \frac{1}{2\sqrt{C_2}} \text{dist}_{G^\Phi}^{B_i}(f_1, f_2),$$

The proof is an adaptation of the one given in [15, section 3.3] for the  $G^A$ -metric.

*Proof.* Let  $f : [0, 1] \rightarrow \text{Imm}$  be a horizontal path, and let  $f_t = a \cdot \nu^f$  denote its derivative. Using the formula from section 3.7 for the variation of the volume we get

$$\partial_t \text{Vol}(f) = - \int_M \text{Tr}(L) a \text{vol}(g) \leq \left| \int_M \text{Tr}(L) a \text{vol}(g) \right|$$

$$\begin{aligned}
&\leq \left( \int_M 1^2 \operatorname{vol}(g) \right)^{\frac{1}{2}} \left( \int_M \operatorname{Tr}(L)^2 a^2 \operatorname{vol}(g) \right)^{\frac{1}{2}} \\
&\leq \sqrt{\operatorname{Vol}(f)} \left( \int_M \frac{\Phi}{C_2} a^2 \operatorname{vol}(g) \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{C_2}} \sqrt{\operatorname{Vol}(f)} \sqrt{G_f^\Phi(f_t, f_t)}.
\end{aligned}$$

Thus

$$\partial_t \sqrt{\operatorname{Vol}(f)} = \frac{\partial_t \operatorname{Vol}(f)}{2\sqrt{\operatorname{Vol}(f)}} \leq \frac{1}{2\sqrt{C_2}} \sqrt{G_f^\Phi(f_t, f_t)}.$$

By integration we get

$$\begin{aligned}
\sqrt{\operatorname{Vol}(f_1)} - \sqrt{\operatorname{Vol}(f_0)} &= \int_0^1 \partial_t \sqrt{\operatorname{Vol}(f)} dt \\
&\leq \int_0^1 \frac{1}{2\sqrt{C_2}} \sqrt{G_f^\Phi(f_t, f_t)} dt = \frac{1}{2\sqrt{C_2}} L_{\operatorname{Imm}}^{G^\Phi}(f).
\end{aligned}$$

Now take the infimum over all horizontal paths  $f$  connecting  $f_1$  and  $f_2$ .  $\square$

**8.5. Lemma** (Second area swept out bound). *For an almost local metric  $G^\Phi$  satisfying*

$$\Phi(\operatorname{Vol}, \operatorname{Tr}(L)) \geq C \operatorname{Vol} \quad \text{with } C > 0$$

*and a horizontal path  $f : [0, 1] \rightarrow \operatorname{Imm}$ , we get the area swept out bound*

$$\sqrt{C} \text{ (area swept out by } f) \leq L_{\operatorname{Imm}}^{G^\Phi}(f),$$

The proof is adapted from proofs for the case of planar curves that can be found in [17, section 3.7], [19, Lemma 3.2], [25, proposition 1] and [24, theorem 7.5].

*Proof.*

$$\begin{aligned}
L_{\operatorname{Imm}}^{G^\Phi}(f) &= \int_0^1 \sqrt{G_f^\Phi(f_t, f_t)} dt = \int_0^1 \left( \int_M \Phi \|f_t\|^2 \operatorname{vol}(g) \right)^{\frac{1}{2}} dt \\
&\geq \sqrt{C} \int_0^1 \sqrt{\operatorname{Vol}(f)} \left( \int_M \|f_t\|^2 \operatorname{vol}(g) \right)^{\frac{1}{2}} dt \geq \sqrt{C} \int_0^1 \int_M 1 \cdot \|f_t\| \operatorname{vol}(g) dt \\
&= \sqrt{C} \int_{[0,1] \times M} \operatorname{vol}(f(\cdot, \cdot)^* \bar{g}) dt = \sqrt{C} \text{ (area swept out by } f). \quad \square
\end{aligned}$$

**8.6. Non-vanishing geodesic distance.** Using the estimates proven above, we get the following result:

**Theorem.** *At least on  $B_e$ , the almost local metric  $G^\Phi$  induces non-vanishing geodesic distance if one of the two following conditions holds:*

- (1)  $\Phi(\operatorname{Vol}, \operatorname{Tr}(L)) \geq C_1 + C_2 \operatorname{Tr}(L)^2$  *for  $C_1, C_2 > 0$ .*
- (2)  $\Phi(\operatorname{Vol}, \operatorname{Tr}(L)) \geq C \operatorname{Vol}$  *for  $C > 0$ .*



**8.7. Fréchet distance and Finsler metric.** The Fréchet distance on shape space  $B_i(M, \mathbb{R}^n)$  is defined as

$$\text{dist}_\infty^{B_i}(F_0, F_1) = \inf_{f_0, f_1} \|f_0 - f_1\|_{L^\infty},$$

where the infimum is taken over all  $f_0, f_1$  with  $\pi(f_0) = F_0, \pi(f_1) = F_1$ . As before,  $\pi$  denotes the projection  $\pi : \text{Imm} \rightarrow B_i$ . Fixing  $f_0$  and  $f_1$ , one has

$$\text{dist}_\infty^{B_i}(\pi(f_0), \pi(f_1)) = \inf_{\varphi} \|f_0 \circ \varphi - f_1\|_{L^\infty},$$

where the infimum is taken over all  $\varphi \in \text{Diff}(M)$ . The Fréchet distance is related to the Finsler metric

$$G^\infty : T \text{Imm}(M, \mathbb{R}^n) \rightarrow \mathbb{R}, \quad h \mapsto \|h^\perp\|_{L^\infty}.$$

**Lemma.** *The path length distance induced by the Finsler metric  $G^\infty$  provides an upper bound for the Fréchet distance:*

$$\text{dist}_\infty^{B_i}(F_0, F_1) \leq \text{dist}_{G^\infty}^{B_i}(F_0, F_1) = \inf_f \int_0^1 \|f_t\|_{G^\infty} dt,$$

where the infimum is taken over all paths

$$f : [0, 1] \rightarrow \text{Imm}(M, \mathbb{R}^n) \quad \text{with} \quad \pi(f(0)) = F_0, \quad \pi(f(1)) = F_1.$$

*Proof.* Since any path  $f$  can be reparametrized such that  $f_t$  is normal to  $f$ , one has

$$\inf_f \int_0^1 \|f_t^\perp\|_{L^\infty} dt = \inf_f \int_0^1 \|f_t\|_{L^\infty} dt,$$

where the infimum is taken over the same class of paths  $f$  as described above. Therefore

$$\begin{aligned} \text{dist}_\infty^{B_i}(F_0, F_1) &= \inf_f \|f(1) - f(0)\|_{L^\infty} = \inf_f \left\| \int_0^1 f_t dt \right\|_{L^\infty} \leq \inf_f \int_0^1 \|f_t\|_{L^\infty} dt \\ &= \inf_f \int_0^1 \|f_t^\perp\|_{L^\infty} dt = \text{dist}_{G^\infty}^{B_i}(F_0, F_1). \end{aligned} \quad \square$$

It is claimed in [12, theorem 13] that  $d_\infty = \text{dist}_{G^\infty}$ . However, the proof given there only works on the vector space  $C^\infty(M, \mathbb{R}^n)$  and not on  $B_i(M, \mathbb{R}^n)$ . The reason is that convex combinations of immersions are used in the proof, but that the space of immersions is not convex.

**8.8. Theorem** (Almost local versus Fréchet distance on shape space). *On the shape space  $B_i(M, \mathbb{R}^n)$  the  $G^\Phi$  distance can not be bounded from below by the Fréchet distance if one of the following conditions holds:*

- (1)  $\Phi \leq C_1 + C_2 \text{Tr}(L)^{2k}$  for  $C_1, C_2 > 0$  and  $k < (\dim(M) + 2)/2$ ,
- (2)  $\Phi \leq C \text{Vol}^k$  for  $C > 0$ ,
- (3)  $\Phi \leq C e^{\text{Vol}}$  for  $C > 0$ ,
- (4)  $\Phi \leq C_1 + C_2 \text{del}(L)^{2l}$  for  $C_1, C_2 > 0$  and  $l < \frac{1}{2} + \frac{1}{\dim(M)}$ .

Indeed, then the identity map

$$\text{Id} : (B_i(M, \mathbb{R}^n), d_{G^\Phi}) \rightarrow (B_i(M, \mathbb{R}^n), d_\infty)$$

is not continuous.

*Proof.* Let  $f_0$  be a fixed immersion of  $M$  into  $\mathbb{R}^n$ , and let  $f_1$  be a translation of  $f_0$  by a vector  $h$  of length  $\ell$ . We will show that the  $H^p$ -distance between  $\pi(f_0)$  and  $\pi(f_1)$  is bounded by a constant  $2L$  that does not depend on  $\ell$ . It follows that the  $H^p$ -distance can not be bounded from below by the Fréchet distance, and this proves the claim.

For small  $r_0$ , we calculate the  $G^\Phi$ -length of the following path of immersions: First scale  $f_0$  by a factor  $r_0$ , then translate it by  $h$ , and then scale it again until it has reached  $f_1$ . The following calculation shows that under one of the above assumption the immersion  $f_0$  can be scaled down to zero in finite  $G^\Phi$ -path length  $L$ .

$$\begin{aligned} L_{\text{Imm}}^{G^\Phi}(r.f_0) &= \\ &= \int_0^1 \sqrt{\int_M \Phi(\text{Vol}(r.f_0), \text{Tr}(L^{r.f_0}), \det(L^{r.f_0})) \bar{g}(r_t.f_0, r_t.f_0) \text{vol}((r.f_0)^*\bar{g})} dt \\ &= \int_0^1 \sqrt{\int_M r_t^2 \cdot \Phi(r^m \text{Vol}(f_0), \frac{1}{r} \text{Tr}(L^{f_0}), \frac{1}{r^m} \det(L^{f_0})) \bar{g}(f_0, f_0) r^m \text{vol}(f_0^*\bar{g})} dt \\ &= \int_1^0 \sqrt{\int_M \Phi(r^m \text{Vol}(f_0), \frac{1}{r} \text{Tr}(L^{f_0}), \frac{1}{r^m} \det(L^{f_0})) \bar{g}(f_0, f_0) r^m \text{vol}(f_0^*\bar{g})} dr \end{aligned}$$

The last integral converges for all of the above assumptions. Scaling down to  $r_0 > 0$  needs even less effort. So we see that the length of the shrinking and growing part of the path is bounded by  $2L$ .

The energy needed for a pure translation of the scaled immersion by distance  $\ell$  is given by ( $f = t.h$ , with  $\bar{g}(h, h) = \ell^2$ ):

$$\begin{aligned} L_{\text{Imm}}^{G^\Phi}(f) &= \int_0^1 \int_M \Phi.\bar{g}(f_t, f_t) \text{vol}(g) dt \\ &= \int_0^1 \int_M \Phi.t^2 \ell^2 \text{vol}(g) dt = \ell^2 \int_M \Phi \text{vol}(g) \\ &= \begin{cases} O(r^{(m-2k)}), & \text{if } \Phi \text{ satisfies (1)} \\ O(r^{m(k+1)}), & \text{if } \Phi \text{ satisfies (2)} \\ O(e^{r.m}.r^m), & \text{if } \Phi \text{ satisfies (3)} \end{cases} \end{aligned}$$

This length tends to zero as  $r$  tends to zero. Therefore

$$\text{dist}_{B_i}^{G^\Phi}(\pi(f_0), \pi(f_1)) \leq \text{dist}_{\text{Imm}}^{G^\Phi}(f_0, f_1) \leq 2L. \quad \square$$

## 9. THE SET OF CONCENTRIC SPHERES

For an almost local metric, the set of spheres with common center  $x \in \mathbb{R}^n$  is a totally geodesic subspace of  $B_i$ . The reason is that it is the fixed point set of a group of isometries acting on  $B_i$ , namely the group of rotations of  $\mathbb{R}^n$  around  $x$ . (We also have to assume uniqueness of solutions to the geodesic equation.) For the

$G^A$  metric and plane curves the set of concentric spheres has been studied in [16] and for Sobolev type metrics they have been studied in [2, 8]. Some work for the  $G^0$ -metric has also been done by [18].

**Theorem.** *Within a set of concentric spheres, any sphere is uniquely described by its radius  $r$ . Thus the geodesic equation within a set of concentric spheres reduces to an ordinary differential equation for the radius. It is given by:*

$$r_{tt} = -r_t^2 \frac{n-1}{\Phi} \left[ \frac{1}{2r} \Phi + \frac{\partial_1 \Phi}{2} \frac{n\pi^{\frac{n}{2}} r^{n-2}}{\Gamma(1 + \frac{n}{2})} + \frac{1}{2r^2} (\partial_2 \Phi) \right].$$

The space of concentric spheres is geodesically complete with respect to a  $G^\Phi$  metric iff

$$\int_0^{r_1} r^{\frac{n-1}{2}} \sqrt{\Phi \left( \frac{n\pi^{\frac{n}{2}} r^{n-1}}{\Gamma(1 + \frac{n}{2})}, -(n-1)/r \right)} dr = \infty \quad r_1 > 0$$

and

$$\int_{r_0}^{\infty} r^{\frac{n-1}{2}} \sqrt{\Phi \left( \frac{n\pi^{\frac{n}{2}} r^{n-1}}{\Gamma(1 + \frac{n}{2})}, -(n-1)/r \right)} dr = \infty \quad r_0 > 0.$$

For the metrics studied in this work, this yields:

$$\begin{aligned} \Phi = \text{Vol}^k &= \frac{n^k \pi^{\frac{kn}{2}}}{\Gamma(1 + \frac{n}{2})^k} r^{k(n-1)} : & \text{incomplete} \\ \Phi = e^{\text{Vol}} &= e^{\frac{n\pi^{\frac{n}{2}}}{\Gamma(1 + \frac{n}{2})}} e^{r^{n-1}} : & \text{incomplete} \\ \Phi = 1 + A \text{Tr}(L)^{2k} &= 1 + A \frac{(n-1)^{2k}}{r^{2k}} : & \text{complete iff } k \geq \frac{n+1}{2} \\ \Phi = \text{Vol}^{\frac{1+n}{1-n}} + A \frac{\text{Tr}(L)^2}{\text{Vol}} &= \frac{C(n)}{r^{n+1}} : & \text{complete.} \end{aligned}$$

*Proof.* The differential equation for the radius can be read off the geodesic equation in section 6.2, when it is taken into account that all functions are constant on each sphere, and that

$$\text{Vol} = \frac{n\pi^{\frac{n}{2}} r^{n-1}}{\Gamma(1 + \frac{n}{2})}, \quad L = -\frac{1}{r} \text{Id}_{TM}, \quad \text{Tr}(L^k) = (-1)^k \frac{n-1}{r^k}.$$

To determine whether the space of concentric spheres is complete, we calculate the length of a path  $f$  connecting a sphere with radius  $r_0$  to a sphere with radius  $r_1$ :

$$\begin{aligned} L_{B_i}^{G^\Phi}(f) &= \int_0^1 \sqrt{G_f^\Phi(f_t^\perp, f_t^\perp)} dt = \\ &= \int_0^1 \sqrt{\int_M \Phi(\text{Vol}, \text{Tr}(L)) r_t^2 \text{vol}(g) dt} \\ &= \int_0^1 |r_t| \sqrt{\Phi \left( \frac{n\pi^{\frac{n}{2}} r^{n-1}}{\Gamma(1 + \frac{n}{2})}, -(n-1)/r \right) \frac{n\pi^{\frac{n}{2}} r^{n-1}}{\Gamma(1 + \frac{n}{2})}} dt \end{aligned}$$

$$= \frac{\sqrt{n\pi}^{\frac{n}{4}}}{\sqrt{\Gamma(1+\frac{n}{2})}} \int_{r_0}^{r_1} r^{\frac{n-1}{2}} \sqrt{\Phi\left(\frac{n\pi^{\frac{n}{2}} r^{n-1}}{\Gamma(1+\frac{n}{2})}, -(n-1)/r\right)} dr. \quad \square$$

## 10. SPECIAL CASES OF ALMOST LOCAL METRICS

**10.1. The  $G^0$ -metric.** The  $G^0$ -metric is the special case of a  $G^\Phi$ -metric with  $\Phi \equiv 1$ . Thus its geodesic equation can be read off from section 5.1. It reads as

$$\begin{aligned} f_t &= a.\nu + Tf.f_t^\top \\ f_{tt} &= -\frac{1}{2} (\|f_t\|^2 \operatorname{Tr}(L).\nu + Tf.\operatorname{grad}^g(\|f_t\|^2)) + (\operatorname{Tr}(L).a - \operatorname{div}^g(hf_t^\top)) .f_t. \end{aligned}$$

We have three conserved quantities, namely:

$$\begin{aligned} g(f_t^\top) \operatorname{vol}(g) &\in \Gamma(T^*M \otimes_M \operatorname{vol}(M)) && \text{reparametrization momentum} \\ \int_M f_t \operatorname{vol}(g) &&& \text{linear momentum} \\ \int_M (f \wedge f_t) \operatorname{vol}(g) &\in \wedge^2 \mathbb{R}^n \cong \mathfrak{so}(n)^* && \text{angular momentum} \end{aligned}$$

The geodesic equation on  $B_i(M, \mathbb{R}^n)$  is well studied. We can read it off from section 6:

$$f_t = a.\nu, \quad a_t = \frac{\operatorname{Tr}(L).a^2}{2}.$$

Sectional curvature is given by

$$R_0(a_1, a_2, a_2, a_1) = \frac{1}{2} \int_M \|a_1 da_2 - a_2 da_1\|_{g^{-1}}^2 \operatorname{vol}(g) \geq 0.$$

This formula is in accordance with [15, section 4.5] since we have codimension one and a flat ambient space, so that only term(6) remains, and for the case of plain curves, it is in accordance with [17, section 3.5].

The  $G^0$ -metric induces vanishing geodesic distance, see section 8.

**10.2. The  $G^A$ -metric.** For a constant  $A > 0$ , the  $G^A$ -metric is defined as

$$G_f^A(h, k) = \int_M (1 + A \operatorname{Tr}(L)^2) \bar{g}(h, k) \operatorname{vol}(g).$$

This metric has been introduced by [16, 15, 17]. It corresponds to an almost local metric  $G^\Phi$  with  $\Phi(x, y, z) = (1 + Ay^2)$ , thus its geodesic equation on  $\operatorname{Imm}(M, \mathbb{R}^n)$

is given by (see section 5.1):

$$\begin{aligned}
 f_t &= a.\nu + Tf.f_t^\top, \\
 f_{tt} &= \frac{1}{2} \left[ -\frac{\Delta((2A \operatorname{Tr}(L)) \|f_t\|^2)}{1 + A \operatorname{Tr}(L)^2} + \|f_t\|^2 \cdot \operatorname{Tr}(L) \left( \frac{2A \operatorname{Tr}(L^2)}{1 + A \operatorname{Tr}(L)^2} - 1 \right) \right] \nu \\
 &\quad + \frac{Tf. \left[ (2A \operatorname{Tr}(L)) \|f_t\|^2 \operatorname{grad}^g(\operatorname{Tr}(L)) - \operatorname{grad}^g((1 + A \operatorname{Tr}(L)^2) \|f_t\|^2) \right]}{2(1 + A \operatorname{Tr}(L)^2)} \\
 &\quad - \left[ \frac{(2A \operatorname{Tr}(L))}{1 + A \operatorname{Tr}(L)^2} (-\Delta a + a \operatorname{Tr}(L^2) + d \operatorname{Tr}(L)(f_t^\top)) \right. \\
 &\quad \left. + \operatorname{div}^g(f_t^\top) - \operatorname{Tr}(L).a \right] f_t.
 \end{aligned}$$

The conserved quantities have the form

$$\begin{aligned}
 (1 + A \operatorname{Tr}(L)^2) g(f_t^\top) \operatorname{vol}(g) &\in \Gamma(T^*M \otimes_M \operatorname{vol}(M)) && \text{reparam. momentum} \\
 \int_M (1 + A \operatorname{Tr}(L)^2) f_t \operatorname{vol}(g) &&& \text{linear momentum} \\
 \int_M (1 + A \operatorname{Tr}(L)^2) (f \wedge f_t) \operatorname{vol}(g) &\in \bigwedge^2 \mathbb{R}^n \cong \mathfrak{so}(n)^* && \text{angular momentum}
 \end{aligned}$$

The horizontal geodesic equation for the  $G^A$  metric reduces to

$$\begin{aligned}
 f_t &= a.\nu \\
 a_t &= \frac{1}{2} a^2 \operatorname{Tr}(L) + \frac{-a^2 A \Delta(\operatorname{Tr}(L)) + 4Aag^{-1}(d \operatorname{Tr}(L), da)}{(1 + A \operatorname{Tr}(L)^2)} \\
 &\quad + \frac{2A \operatorname{Tr}(L) \|da\|_{g^{-1}}^2 - A \operatorname{Tr}(L) \operatorname{Tr}(L^2) a^2}{(1 + A \operatorname{Tr}(L)^2)}
 \end{aligned}$$

For the case of curves immersed in  $\mathbb{R}^2$ , this formula specializes to the formula given in [16, section 4.2]. (When verifying this, remember that  $\Delta = -D_s^2$  in the notation of [16].)

The curvature tensor  $R_0(a_1, a_2, a_1, a_2)$  is the sum of:

$$\boxed{P_1 P} \quad \boxed{Q_3 Q_3} \quad \text{negative terms,}$$

$$\boxed{P_6 P} \quad \boxed{Q_5 Q_5} \quad \text{positive terms, and}$$

$P_3P$   $Q_1Q_5$   $Q_3Q_5$  indefinite terms.

$$\begin{aligned}
R_0(a_1, a_2, a_1, a_2) = & \int_M A(a_1 \Delta a_2 - a_2 \Delta a_1)^2 \text{vol}(g) \\
& + \int_M 2A \text{Tr}(L) g_2^0((a_1 da_2 - a_2 da_1) \otimes (a_1 da_2 - a_2 da_1), s) \text{vol}(g) \\
& + \int_M \frac{1}{1 + A \text{Tr}(L)^2} \left[ -4A^2 g^{-1}(d \text{Tr}(L), a_1 da_2 - a_2 da_1)^2 \right. \\
& - \left( \frac{1}{2} (1 + A \text{Tr}(L)^2)^2 + 2A^2 \text{Tr}(L) \Delta(\text{Tr}(L)) + 2A^2 \text{Tr}(L^2) \text{Tr}(L)^2 \right) \\
& \quad \cdot \|a_1 da_2 - a_2 da_1\|_{g^{-1}}^2 + (2A^2 \text{Tr}(L)^2) \|da_1 \wedge da_2\|_{g_0^2}^2 \\
& \left. + (8A^2 \text{Tr}(L)) g_2^0(d \text{Tr}(L) \otimes (a_1 da_2 - a_2 da_1), da_1 \wedge da_2) \right] \text{vol}(g)
\end{aligned}$$

We want to express the curvature in terms of the basic skew symmetric forms. Therefore, mimicking the notation of [16, 17] we define

$$W_2 = a_1 da_2 - a_2 da_1, \quad W_{22} = a_1 \Delta a_2 - a_2 \Delta a_1, \quad W_{12} = da_1 \wedge da_2.$$

Then the above equation reads as:

$$\begin{aligned}
R_0(a_1, a_2, a_1, a_2) = & \int_M A W_{22}^2 \text{vol}(g) + \int_M 2A \text{Tr}(L) g_2^0(W_2 \otimes W_2, s) \text{vol}(g) \\
& + \int_M \frac{1}{1 + A \text{Tr}(L)^2} \left[ -4A^2 g^{-1}(d \text{Tr}(L), W_2)^2 \right. \\
& - \left( \frac{1}{2} (1 + A \text{Tr}(L)^2)^2 + 2A^2 \text{Tr}(L) \Delta(\text{Tr}(L)) + 2A^2 \text{Tr}(L^2) \text{Tr}(L)^2 \right) \|W_2\|_{g^{-1}}^2 \\
& \left. + (2A^2 \text{Tr}(L)^2) \|W_{12}\|_{g_0^2}^2 + (8A^2 \text{Tr}(L)) g_2^0(d \text{Tr}(L) \otimes W_2, W_{12}) \right] \text{vol}(g)
\end{aligned}$$

For the case of plain curves, this formula specializes to the formula given in [17, section 3.6].

The  $G^A$ -metric satisfies condition (1) from section 8.6, thus it induces non-vanishing geodesic distance.

**10.3. Conformal metrics.** The conformal metrics correspond to almost local metrics  $G^\Phi$  with  $\Phi = \Phi(\text{Vol})$ . For the case of planar curves these metrics have been treated in [23, 24, 25, 19]. [19] provides very interesting estimates on geodesic distance induced by metrics with  $\Phi(\text{Vol}) = \text{Vol}$  and  $e^{\text{Vol}}$ . The geodesic equation on

$\text{Imm}(M, \mathbb{R}^n)$  is given by:

$$\begin{aligned} f_t &= h = a.\nu + Tf.h^\top, \\ h_t &= -\frac{1}{2} \left[ \frac{\Phi'}{\Phi} \left( \int_M \|h\|^2 \text{vol}(g) \right) \text{Tr}(L).\nu \right. \\ &\quad \left. + \|h\|^2 \text{Tr}(L).\nu + Tf.\text{grad}^g(\|h\|^2) \right] \\ &\quad + \left[ \frac{\Phi'}{\Phi} \left( \int_M \text{Tr}(L).a \text{vol}(g) \right) + \text{Tr}(L).a - \text{div}^g(h^\top) \right].h \end{aligned}$$

The conserved quantities are given by

$$\begin{aligned} \Phi(\text{Vol})g(f_t^\top) \text{vol}(g) &\in \Gamma(T^*M \otimes_M \text{vol}(M)) && \text{reparam. momentum} \\ \Phi(\text{Vol}) \int_M f_t \text{vol}(g) &&& \text{linear momentum} \\ \Phi(\text{Vol}) \int_M (f \wedge f_t) \text{vol}(g) &\in \wedge^2 \mathbb{R}^n \cong \mathfrak{so}(n)^* && \text{angular momentum} \end{aligned}$$

The horizontal part of the geodesic equation is given by

$$\begin{aligned} a_t &= \bar{g} \left( \frac{1}{2} H(a.\nu, a.\nu) - K(a.\nu, a.\nu), \nu \right) \\ &= -\frac{\Phi'}{2\Phi} \left( \int_M a^2 \text{vol}(g) \right) \text{Tr}(L) + \frac{1}{2} a^2 \text{Tr}(L) + \frac{\Phi'}{\Phi} \left( \int_M a. \text{Tr}(L) \text{vol}(g) \right) a. \end{aligned}$$

To simplify this equation let  $b(t) = \Phi(\text{Vol}).a(t)$ . We get

$$\begin{aligned} b_t &= \Phi'.(D_{(f,a.\nu)} \text{Vol}).a + \Phi.a_t \\ &= -\Phi'.a. \int_M \text{Tr}(L).a. \text{vol}(g) + \Phi \frac{1}{2} a^2. \text{Tr}(L) \\ &\quad - \frac{1}{2} \Phi' \left( \int_M a^2 \text{vol}(g) \right). \text{Tr}(L) + \Phi'.a. \int_M \text{Tr}(L).a \text{vol}(g) \\ &= -\frac{1}{2} \Phi' \int_M a^2 \text{vol}(g). \text{Tr}(L) + \frac{1}{2} \Phi a^2. \text{Tr}(L). \end{aligned}$$

Thus the geodesic equation of the conformal metric  $G^\Phi$  on  $B_i$  is

$$\begin{aligned} f_t &= \frac{b(t)}{\Phi(\text{Vol})} \nu \\ b_t &= \frac{\text{Tr}(L)}{2\Phi(\text{Vol})} \left( b^2 - \frac{\Phi'(\text{Vol})}{\Phi(\text{Vol})} \int_M b^2 \text{vol}(g) \right). \end{aligned}$$

For the case of curves immersed in  $\mathbb{R}^2$ , this formula specializes to the formula given in [17, section 3.7].

Assuming that  $\Phi'$  and  $\Phi''$  are non-negative, the curvature tensor consists of the following summands.

$$\boxed{P_4 P} \quad \boxed{Q_2 Q_2} \quad \text{are the positive summands.}$$

$$\boxed{P_1 P} \quad \boxed{Q_4 Q_4} \quad \boxed{Q_1 Q_2} \quad \text{are the negative summands.}$$

$\boxed{Q_2 Q_4}$  is indefinite, but assuming that  $\frac{\Phi'}{\Phi}$  is a non-negative constant, it is negative. Solving the ODE  $\frac{\Phi'}{\Phi} = C > 0$  leads to  $\Phi(\text{Vol}) = e^{C \cdot \text{Vol}}$ . In the case of curves, conformal metrics of this type have been studied by [12] and [19].

$\boxed{P_2 P}$  is indefinite.

Since the formula for sectional curvature with general  $\Phi = \Phi(\text{Vol})$  is still too long, we will only print the formula for  $\Phi(\text{Vol}) = \text{Vol}$ . To shorten notation we will write  $\bar{a}$  for the integral over  $a \in C^\infty(M)$ , i.e.

$$\bar{a} = \int_M a \, \text{vol}(g).$$

Then the sectional curvature reads as:

$$\begin{aligned} R_0(a_1, a_2, a_1, a_2) = & -\frac{1}{2} \text{Vol} \int_M \|a_1 da_2 - a_2 da_1\|_{g^{-1}}^2 \text{vol}(g) \\ & + \frac{1}{4 \text{Vol}} \overline{\text{Tr}(L)^2} \left( \overline{a_1^2 \cdot a_2^2} - \overline{a_1 \cdot a_2^2} \right) \\ & + \frac{1}{4} \left( \overline{a_1^2 \cdot \text{Tr}(L)^2 a_2^2} - 2 \overline{a_1 \cdot a_2} \cdot \overline{\text{Tr}(L)^2 a_1 \cdot a_2} + \overline{a_2^2 \cdot \text{Tr}(L)^2 a_1^2} \right) \\ & - \frac{3}{4 \text{Vol}} \left( \overline{a_1^2 \cdot \text{Tr}(L) a_2^2} - 2 \overline{a_1 \cdot a_2} \cdot \overline{\text{Tr}(L) a_1 \cdot \text{Tr}(L) a_2} + \overline{a_2^2 \cdot \text{Tr}(L) a_1^2} \right) \\ & + \frac{1}{2} \left( \overline{a_1^2 \cdot \text{Tr}^g((da_2)^2)} - 2 \overline{a_1 \cdot a_2} \cdot \overline{\text{Tr}^g(da_1 \cdot da_2)} + \overline{a_2^2 \text{Tr}^g((da_1)^2)} \right) \\ & - \frac{1}{2} \left( \overline{a_1^2 \cdot a_2^2 \cdot \text{Tr}(L^2)} - 2 \overline{a_1 \cdot a_2} \cdot \overline{a_1 \cdot a_2 \cdot \text{Tr}(L^2)} + \overline{a_2^2 \cdot a_1^2 \cdot \text{Tr}(L^2)} \right). \end{aligned}$$

For the case of curves immersed in  $\mathbb{R}^2$ , this formula is in accordance with the formula given in [17, section 3.7].

From Condition (2) in section 8.6 we read off that the conformal metrics induce non-vanishing geodesic distance if  $\Phi(\text{Vol}) \geq C \cdot \text{Vol}$  for some constant  $C > 0$ .

**10.4. A scale invariant metric.** For a constant  $A > 0$  we define the metric

$$G_f^{SI}(h, k) = \int_M \left( \text{Vol}^{\frac{1+n}{1-n}} + A \frac{\text{Tr}(L)^2}{\text{Vol}} \right) \bar{g}(h, k) \, \text{vol}(g).$$

Scale invariance means that this metric does not change when  $f, h, k$  are replaced by  $\lambda f, \lambda h, \lambda k$  for  $\lambda > 0$ . To see that  $G^{SI}$  is scale invariant, we calculate as in [17] how the scaling factor  $\lambda$  changes the metric, volume form, volume and mean curvature. We fix an oriented chart  $(u^1, \dots, u^{n-1})$  on  $M$ . Then

$$\begin{aligned} (\lambda f)^* \bar{g}(\partial_i, \partial_j) &= \bar{g}(T(\lambda f) \cdot \partial_i, T(\lambda f) \cdot \partial_j) = \lambda^2 \cdot f^* \bar{g}(\partial_i, \partial_j) \\ \text{vol}((\lambda \cdot f)^* \bar{g}) &= \sqrt{\det(\lambda^2 (f^* \bar{g})|_U)} \, du^1 \wedge \dots \wedge du^{n-1} = \lambda^{n-1} \text{vol}(f^* \bar{g}) \\ \text{Tr}(L((\lambda f)^* \bar{g})) &= ((\lambda f)^* \bar{g})^{ij} \bar{g} \left( \frac{\partial^2 (\lambda f)}{\partial_i \partial_j}, \nu^{\lambda \cdot f} \right) \\ &= \frac{\lambda}{\lambda^2} (f^* \bar{g})^{ij} \bar{g} \left( \frac{\partial^2 f}{\partial_i \partial_j}, \nu^f \right) = \frac{1}{\lambda} \text{Tr}(L(f)). \end{aligned}$$



The scale invariance of the metric  $G^{SI}$  follows. Thus along geodesics we have an additional conserved quantity (see section 5.2), namely:

$$\boxed{\int_M \left( \text{Vol}^{\frac{1+n}{1-n}} + A \frac{\text{Tr}(L)^2}{\text{Vol}} \right) \bar{g}(f, f_t) \text{vol}(g) \quad \text{scaling momentum}}$$

From 6 we can read off the geodesic equation for  $G^{SI}$  on  $B_i$ :

$$\boxed{\begin{aligned} f_t &= a \cdot \nu, \\ a_t &= \frac{1}{2} a^2 \text{Tr}(L) + \frac{1}{\text{Vol}^{\frac{1+n}{1-n}} + A \frac{\text{Tr}(L)^2}{\text{Vol}}} \\ &\quad \left[ -\frac{1}{2} \text{Tr}(L) \int_M \left( \frac{1+n}{1-n} \text{Vol}^{\frac{2n}{1-n}} - A \frac{\text{Tr}(L)^2}{\text{Vol}^2} \right) a^2 \text{vol}(g) - A \frac{\Delta(\text{Tr}(L)) \cdot a^2}{\text{Vol}} \right. \\ &\quad + \frac{4A \cdot a}{\text{Vol}} g^{-1}(d \text{Tr}(L), da) + \frac{2A \text{Tr}(L)}{\text{Vol}} \|da\|_{g^{-1}}^2 \\ &\quad \left. + \left( \frac{1+n}{1-n} \text{Vol}^{\frac{2n}{1-n}} - A \frac{\text{Tr}(L)^2}{\text{Vol}^2} \right) a \int_M \text{Tr}(L) \cdot a \text{vol}(g) - A \frac{\text{Tr}(L^2) \text{Tr}(L)}{\text{Vol}} a^2 \right]. \end{aligned}}$$

For the case of curves immersed in  $\mathbb{R}^2$ , this formula specializes to the formula given in [17, section 3.8]. (When verifying this, remember that  $\Delta = -D_s^2$  in the notation of [17].)

The metric  $G^{SI}$  induces non-vanishing geodesic distance. This follows from the fact that  $\log(\text{Vol})$  is Lipschitz, see [17, section 3.8].

## 11. NUMERICAL RESULTS

**11.1. Discretizing the horizontal path energy.** We want to solve the boundary value problem for geodesics in shape space of surfaces in  $\mathbb{R}^3$  with respect to several almost local metrics, more specifically with respect to  $G^\Phi$ -metrics with

$$\Phi = \text{Vol}^k, \quad \Phi = e^{\text{Vol}}, \quad \Phi = 1 + A \text{Tr}(L)^{2k}$$

and the scale-invariant metric

$$\Phi = \text{Vol}^{\frac{1+3}{1-3}} + A \frac{\text{Tr}(L)^2}{\text{Vol}}.$$

In order to solve this infinite-dimensional problem numerically, we will reduce it to a finite-dimensional problem by approximating an immersed surface by a triangular mesh.

One approach to solve the boundary value problem is by the method of geodesic shooting. This method is based on iteratively solving the initial value problem for geodesics while suitably adapting the initial conditions.

Another approach, and the approach we will follow, is to minimize horizontal path energy

$$E^{\text{hor}}(f) = \int_0^1 \int_M \Phi(\text{Vol}, \text{Tr}(L)) \bar{g}(f_t, \nu)^2 \text{vol}(g)$$

over the set of paths  $f$  of immersions with fixed endpoints. Note that by definition, the horizontal path energy does not depend on reparametrizations of the surface.

Nevertheless we want the triangular mesh to stay regular. This can be achieved by adding a penalty functional to the horizontal path energy.

**11.2. Discrete path energy.** To discretize the horizontal path energy

$$E^{\text{hor}}(f) = \int_0^1 \int_M \Phi(\text{Vol}, \text{Tr}(L)) \bar{g}(f_t, \nu)^2 \text{vol}(g),$$

one has to find a discrete version of all the involved terms, notably the mean curvature. We will follow [20] to do this. Let  $V, E, F$  denote the vertices, edges and faces of the triangular mesh, and let  $\text{star}(p)$  be the set of faces surrounding vertex  $p$ . Then the discrete mean curvature at vertex  $p$  can be defined as

$$\text{Tr}(L)(p) = \frac{\|\text{vector mean curvature}\|}{\|\text{vector area}\|} = \frac{\|\nabla_p(\text{surface area})\|}{\|\nabla_p(\text{enclosed volume})\|}.$$

Here  $\nabla_p$  stands for a discrete gradient, and

$$(\text{vector mean curvature})_p = \nabla_p(\text{surface area}) = \sum_{(p, p_i) \in E} (\cot \alpha_i + \cot \beta_i)(p - p_i)$$

is the vector mean curvature defined by the cotangent formula. In this formula,  $\alpha_i$  and  $\beta_i$  are the angles opposite the edge  $(p, p_i)$  in the two adjacent triangles. For the numerical simulation it is advantageous to express this formula in terms of scalar and cross products instead of the cotangents. Furthermore,

$$(\text{vector area})_p = \nabla_p(\text{enclosed volume}) = \sum_{f \in \text{Star}(p)} \nu(f) \cdot (\text{surface area of } f)$$

is the vector area at vertex  $p$ .

We discretize the time by

$$0 = t_1 < \dots < t_{N+1} = 1.$$

Then the  $(N-1)(\#V)$  free variables representing the path of immersions  $f$  are

$$f(t_i, p), \quad \text{with } 2 \leq i \leq N, \quad p \in V.$$

$f(0, p)$  and  $f(1, p)$  are not free variables, since they define the fixed boundary shapes.  $f_t$  can be approximated by either forward increments

$$f_t^{fw}(t_i, p) = \frac{f(t_{i+1}, p) - f(t_i, p)}{t_{i+1} - t_i}$$

or backward increments

$$f_t^{bw}(t_i, p) = \frac{f(t_i, p) - f(t_{i-1}, p)}{t_i - t_{i-1}}.$$

We use a combination of both to make path energy symmetric. (Instead of this we could have also used the central difference quotient. However minimizing an energy functional depending on central differences favors oscillations, since they are not felt by the central differences.) Using the discrete definitions of normal vector and

increments we can calculate  $f_t^\perp$  at every vertex  $p$  and are now able to write down the discrete horizontal path energy:

$$\begin{aligned} G_f^\Phi(h^\perp, k^\perp) &= \sum_{p \in V} \sum_{F \ni p} \Phi\left(\text{Vol}, \text{Tr}(L)(p)\right) \\ &\quad \cdot \bar{g}(h(p), \nu(F)) \cdot \bar{g}(k(p), \nu(F)) \frac{\text{area}(\text{star}^f(p))}{3} \\ E^{\text{hor}}(f) &= \sum_{i=1}^N \frac{t_{i+1} - t_i}{2} \\ &\quad \cdot \left( G_{f(t_i)}^\Phi(f_t^{fw}(t_i), f_t^{fw}(t_i)) + G_{f(t_{i+1})}^\Phi(f_t^{bw}(t_{i+1}), f_t^{bw}(t_{i+1})) \right). \end{aligned}$$

This is not the only way to discretize the energy functional. There are several ways to distribute the discrete energy on faces, vertices and edges. Depending on how this was done, the minimizer converged faster, slower or even not at all. However if the minimizer converged to a smooth solution, the results were qualitatively the same. This increased our belief in the discretization. However we do not guarantee the accuracy of the simulations in this section.

This energy functional does not depend on the parametrization of the surface at each instant of time. So we are free to choose a suitable parametrization. We do this by adding to the energy functional a term penalizing irregular meshes. So instead of minimizing horizontal path energy, we minimize the sum of horizontal path energy and a penalty term. The penalty term measures the deviation of angles from the “perfect angle”  $2\pi$  divided by the number of surrounding triangles, i.e.

$$\sum_{t=2}^N \sum_{p \in V} \sum_{(p,q,r) \in \Delta} \left| \angle(pq, pr) - (\text{perfect angle}) \right|^k, \quad k \in \mathbb{N}.$$

**11.3. Numerical implementation.** Discrete path energy depends on a very high number of real variables, namely three times the number of vertices times one less than the number of time steps. In the numerical experiments that we have done, this were between 5.000 and 50.000 variables. To solve this problem we used the nonlinear solver IPOPT (Interior Point OPTimizer [22]). IPOPT uses a filter based line search method to compute the minimum. In this process it needs the gradient and the Hessian of the energy. IPOPT was invoked by AMPL (A Modeling Language for Mathematical Programming [6]). The advantage of using AMPL is that it is able to automatically calculate the gradient and Hessian. The user only has to write a model and data file for AMPL in a quite readable notation. The data file containing the definition of the combinatorics of the triangle mesh was automatically generated by the computer algebra system Mathematica. As an example, some discretizations of the sphere that we used can be seen in figure 1.

**11.4. Scaling a sphere.** In section 9 we studied the set of concentric spheres in  $n$  dimensions. In dimension three the geodesic equation for the radius simplifies to

$$r_{tt} = -r_t^2 \frac{1}{\Phi} \left[ \frac{1}{r} \Phi + \partial_1 \Phi 4r^2 \pi + \frac{1}{r^2} (\partial_2 \Phi) + \frac{1}{r^2} (\partial_3 \Phi) \right].$$

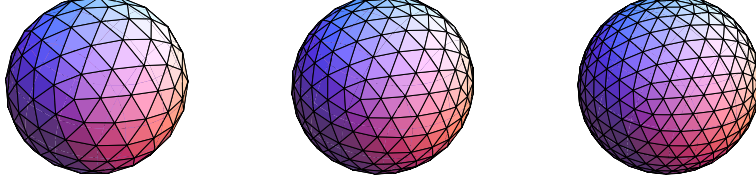


FIGURE 1. Triangulations of a sphere with 320, 500 and 720 triangles, respectively.

*This equation is in accordance with the numerical results obtained by minimizing the discrete path energy defined in section 11.2. As will be seen, the numerics show that the shortest path connecting two concentric spheres in fact consists of spheres with the same center, and that the above differential equation is (at least qualitatively) satisfied. Furthermore, in our experiments the optimal paths obtained were independent of the initial path used as a starting value for the optimization.*

In all numerical experiments of this section we used 50 timesteps and a triangulation with 320 triangles.

For conformal metrics of the type  $\Phi = \text{Vol}^k$  and  $\Phi = e^{\text{Vol}}$ , the differential equation for the radius is:

$$\begin{aligned} \Phi = \text{Vol}^k : \quad & r_{tt} = -r_t^2 \frac{k+1}{r}, \\ \Phi = e^{\text{Vol}} : \quad & r_{tt} = -r_t^2 \left( \frac{1}{r} + 4r\pi \right). \end{aligned}$$

Note that the equation for  $\Phi = \text{Vol}^{-1}$  is  $r_{tt} = 0$ . These equations have explicit analytic solutions given by

$$\begin{aligned} \Phi = \text{Vol}^k : \quad & r = C_1 ((k+2)t - C_2)^{\frac{1}{k+2}} \\ \Phi = e^{\text{Vol}} : \quad & r = \frac{1}{2\pi} \sqrt{\log(C_1 t + C_2)}. \end{aligned}$$

A comparison of the numerical results with the exact analytic solutions can be seen in figure 2 and 3. The solid lines are the exact solutions. For the numerical solutions, 50 time steps and a triangulation with 320 triangles (see figure 1) were used. Note that for big radii as in figure 2, the solution for  $\Phi = e^{\text{Vol}}$  has a very steep ascent, is more curved and lies above the solutions for  $\Phi = \text{Vol}, \text{Vol}^2, \text{Vol}^3$ . For small radii, it lies below these solutions, as can be seen in figure 3. Note also that when the ascent gets too steep, the discrete solution is somewhat inexact as in figure 2.

For mean curvature weighted metrics, the differential equation for the radius is:

$$\Phi = 1 + A \text{Tr}(L)^{2k} : \quad r_{tt} = -r_t^2 \left( \frac{1}{r} - \frac{2kA2^{2k-1}}{r^{2k+1} + A2^{2k}r} \right).$$

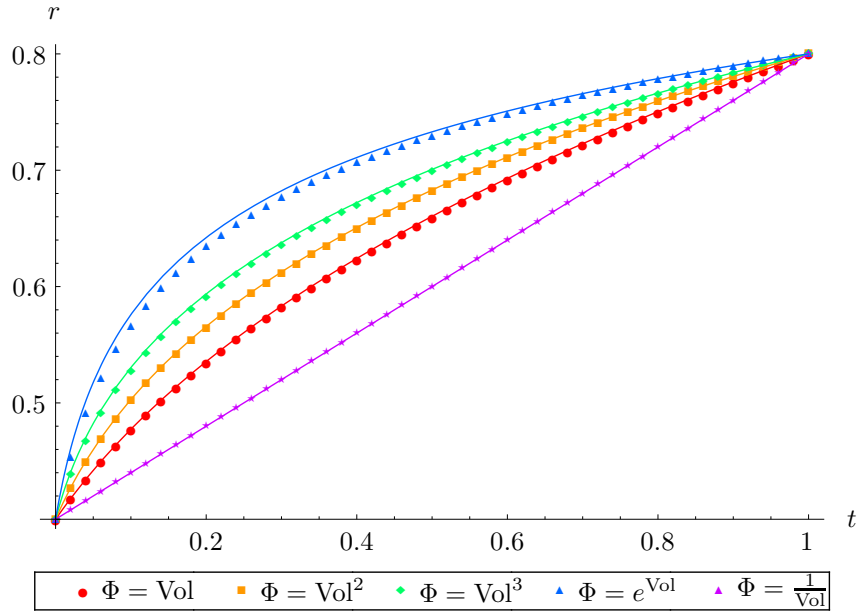


FIGURE 2. Geodesics between concentric spheres of radius 0.4 to 0.8 for several conformal metrics. Solid lines are the exact solutions.

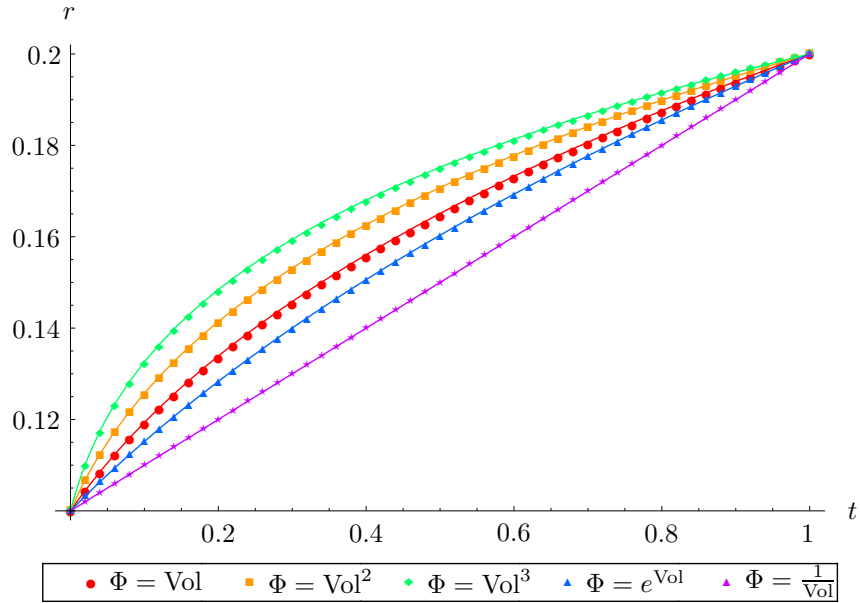


FIGURE 3. Geodesics between concentric spheres of radius 0.1 to 0.2 for several conformal metrics. Solid lines are the exact solutions.

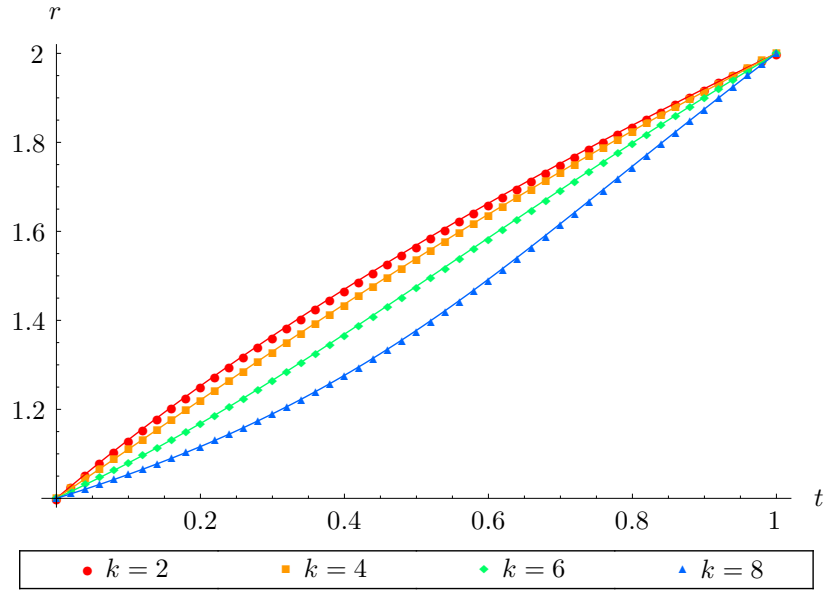


FIGURE 4. Geodesics between concentric spheres for  $\Phi = 1 + 0.1 \text{Tr}(L)^k$ , and varying  $k$ . Solid lines are the exact solutions.

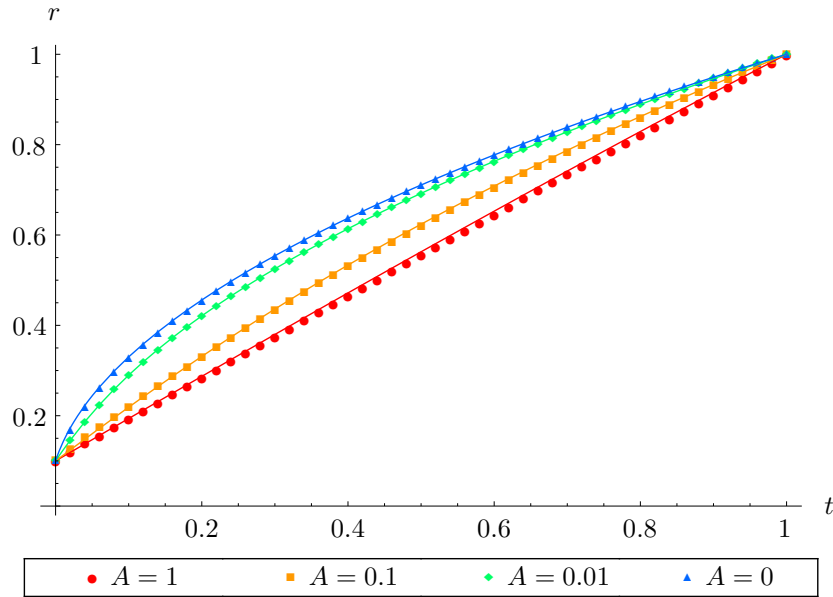


FIGURE 5. Geodesics between concentric spheres for  $\Phi = 1 + A \text{Tr}(L)^2$  and varying  $A$ . Solid lines are the exact solutions.

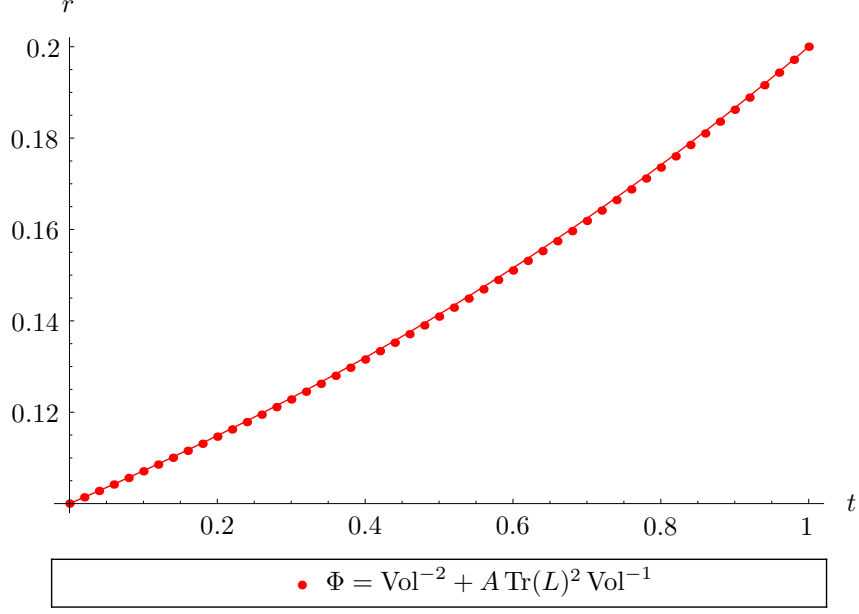


FIGURE 6. Geodesics between concentric spheres for the scale-invariant metric.

The numerics for these metrics are shown in figure 4 and figure 5. Note that we got convergence to a path consisting of concentric spheres even for the  $G^0$ -metric ( $A = 0$ ), even though we know from the theory that this is not the shortest path. In fact, there are no shortest paths for the  $G^0$  metric since it has vanishing geodesic distance [15].

For the scale-invariant metric, the differential equation is given by:

$$\Phi = \text{Vol}^{-2} + A \frac{\text{Tr}(L)^2}{\text{Vol}} : \quad r_{tt} = \frac{r_t^2}{r}.$$

This equation has an explicit analytical solution:

$$\Phi = \text{Vol}^{-2} + A \frac{\text{Tr}(L)^2}{\text{Vol}} : \quad r = C_1 e^{C_2 t}.$$

Note that this equation and therefore its solution is independent of  $A$ . Again, this is confirmed by the numerics, see figure 6.

**11.5. Translation of a sphere.** In this section we will study geodesics between a sphere and a translated sphere for various almost local metrics of the type  $\Phi = \text{Vol}^k$ ,  $\Phi = e^{\text{Vol}}$  and  $\Phi = 1 + A \text{Tr}(L)^{2k}$ .

Depending on the distance (relative to the radius) of the two translated spheres, different behaviors can be observed.

**High distance:**

- *Shrink and grow:* For some metrics it is possible to shrink a sphere in finite time to zero. For these metrics long translation goes via a shrinking and growing part. Metrics with this behavior are:  $\Phi = \text{Vol}^k$ ,  $\Phi = e^{\text{Vol}}$  and  $\Phi = 1 + A \text{Tr}(L)^2$ . This phenomenon is studied in more detail in section 11.6, see also figure 8.
- *Moving an optimal middle shape:* For some of the metrics translation of a sphere with a certain optimal radius is a geodesic. For these metrics geodesics for long translations scale the sphere to the optimal radius and translate the sphere with the optimal radius. Metrics with this behavior are  $\Phi = 1 + A \text{Tr}(L)^{2k}$  for  $k > 1$ . This behavior is studied in section 11.7.

#### Low distance:

- Geodesics of pure translation. ( $\Phi = 1 + A \text{Tr}(L)^{2k}$  for  $k > 1$ , c.f. figure 11)
- Geodesics that pass through an ellipsoid, where the longer principal axis is in the direction of the translation (Conformal metrics, c.f. figure 7).
- Geodesics that pass through an ellipsoid, where the principal axis in the direction of the translation is shorter ( $\Phi = 1 + A \text{Tr}(L)^{2k}$  for  $k > 1$ , c.f. figure 11).
- Geodesics that pass through a cigar shaped figure ( $\Phi = 1 + A \text{Tr}(L)^2$ , c.f. figure 10.)

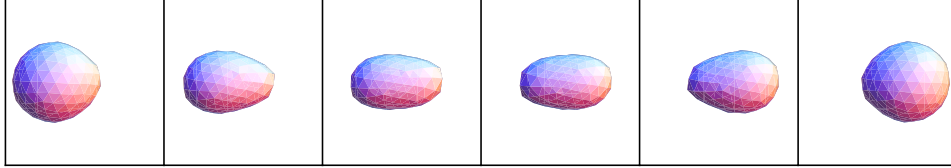


FIGURE 7. Geodesic between two unit spheres translated by distance 1.5 for  $\Phi = \text{Vol}$ . 20 timesteps and a triangulation with 500 triangles were used. Time progresses from left to right. Boundary shapes  $t = 0$  and  $t = 1$  are not included.

**11.6. Shrink and grow.** In section 9 we showed that it is possible to shrink a sphere to zero in finite time for some of the metrics, namely conformal metrics with  $\Phi = \text{Vol}^k$  or  $\Phi = e^{\text{Vol}}$  and for the  $G^A$  metric. For these metrics geodesics of long translation will go via a shrinking and growing part, and almost all of the translation will be done with the shrunken version of the shape. An example of such a geodesic can be seen in figure 8.

We could not determine numerically whether a collapse of the sphere to a point occurs or not. But the more time steps were used, the smaller the ellipsoid in the middle turned out. Also, the energy of the geodesic path comes very close to the energy needed to shrink the sphere to a point and blow it up again. It is remarkable that almost all of the translation is concentrated at a single time step, independently of the number of timesteps that were used. The reason for this behavior is that high volumes are penalized so much: In the case of figure 8,  $e^{\text{Vol}}$  is more than 1000 times smaller in the middle than at the boundary shapes.





FIGURE 8. Geodesic between two unit spheres translated by distance 2 for  $\Phi = e^{\text{Vol}}$ . 20 timesteps and a triangulation with 500 triangles were used. Time progresses from left to right. Boundary shapes  $t = 0$  and  $t = 1$  are not included.

We now want find out under what conditions on the distance and radius of the boundary spheres of the geodesic this behavior can occur. To do this, we compare the energy needed for a pure translation with the energy needed to first shrink the sphere to almost zero, then move it, and then blow it up again.

The energy needed for a pure translation of a sphere with radius  $r$  by distance  $\ell$  in the direction of a unit vector  $e_1$  is given by

$$\begin{aligned}
 E &= \int_0^1 \int_{S^2} \Phi(\text{Vol}, \text{Tr}(L)) \bar{g}(\ell \cdot e_1, \nu)^2 \text{vol}(g) dt \\
 &= \Phi(4r^2\pi, -\frac{2}{r}) \int_0^\pi \int_0^{2\pi} \bar{g}\left(\ell \cdot e_1, \begin{pmatrix} \cos \varphi \sin \theta \\ \sin \varphi \sin \theta \\ \cos \theta \end{pmatrix}\right)^2 r^2 \sin \theta d\varphi d\theta \\
 &= \Phi(4r^2\pi, -\frac{2}{r}) \int_0^\pi \int_0^{2\pi} \ell^2 \cdot (\cos \varphi \sin \theta)^2 r^2 \sin \theta d\varphi d\theta = \Phi(4r^2\pi, -\frac{2}{r}) \cdot \frac{4\pi}{3} \ell^2 \cdot r^2
 \end{aligned}$$

Any other unit vector can be chosen instead of  $e_1$ , yielding the same result.

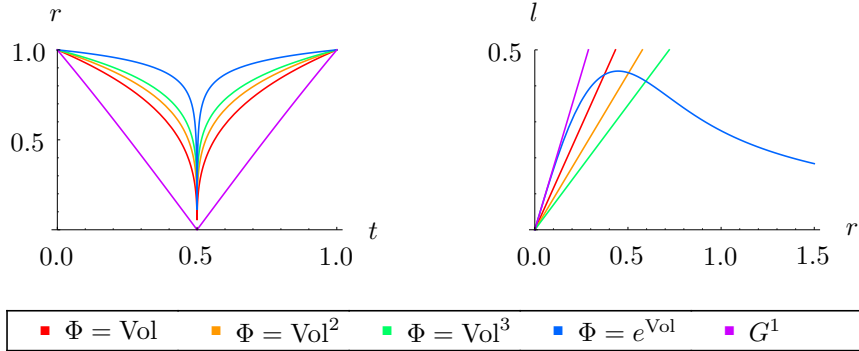


FIGURE 9. Left: Shrinking a sphere to zero along a geodesic path and blowing it up again. Right: Pairs of  $\ell$  and  $r$  such that translating a sphere of radius  $r$  by distance  $\ell$  needs as much energy as shrinking it to zero and blowing it up again.  $G^1$  stands for the  $G^A$  metric with  $A = 1$ .

We will now calculate the energy needed for shrinking the sphere, moving it, and blowing it up again. The energy needed for translating a sphere of radius almost

zero can be neglected. Shrinking and blowing up is done using the solutions to the geodesic equation for the radius from the last section, where one has to adapt the constants to the boundary conditions. For the shrinking part, we have  $r(0) = r$  and  $r(\frac{1}{2}) = 0$ , and for the growing part we have  $r(\frac{1}{2}) = 0, r(1) = r$ , see figure 9 (left).

The energy of the path is

$$\begin{aligned} \Phi = \text{Vol}^k : \quad E &= \int_0^1 \text{Vol}^k \int_{S^2} r_t^2 \text{vol}(g) dt = \frac{4^{k+2} \pi^{k+1}}{(k+2)^2} r^{2k+4} \\ \Phi = e^{\text{Vol}} : \quad E &= \int_0^1 e^{\text{Vol}} \int_{S^2} r_t^2 \text{vol}(g) dt = \frac{1}{\pi} (e^{2\pi r^2} - 1)^2. \end{aligned}$$

The energy of the two different paths are the same when

$$\begin{aligned} \Phi = \text{Vol}^k : \quad \ell &= \frac{2\sqrt{3}r}{k+2} \\ \Phi = e^{\text{Vol}} : \quad \ell &= \frac{\sqrt{3}(1 - e^{-2\pi r^2})}{2r\pi}. \end{aligned}$$

These curves are shown in figure 9 (right). We did not derive an analytic solution for the  $G^A$  metric, but for  $A = 1$  one can see the solution curves in figure 9.

**11.7. Moving an optimal shape.** In the following we want to determine whether pure translation of a sphere is a geodesic. Therefore let  $f_t = f_0 + b(t) \cdot e_1$ , where  $f_0$  is a sphere of radius  $r$  and where  $b(t)$  is constant on  $M$ . Plugging this into the geodesic equation from section 5.1 yields an ODE for  $b(t)$  and a part which has to vanish identically. The latter is given by:

$$(1) \quad (\partial_1 \Phi) \frac{2}{r} 4r^2 \pi + (\partial_2 \Phi) \frac{2}{r^2} + \Phi \frac{2}{r} = 0$$

For conformal metrics this equation is only satisfied if  $\Phi = \text{Vol}^{-1}$ . Since this metric induces vanishing geodesic distance (see section 8) we are not interested in this case. For curvature weighted metrics the above equation reads as:

$$\Phi = 1 + A \text{Tr}(L)^{2k} : \quad \frac{4^k A(k-1)}{r^{4k}} = 1$$

Solutions to these equations are given by:

$$\Phi = 1 + A \text{Tr}(L)^{2k} : \quad r = 2^{\frac{2k}{k-1}} \sqrt{A(k-1)}, \quad k \geq 1.$$

For the most prominent example the  $G^A$  metric this yields  $r = 0$  and therefore translation can never be a geodesic for this type of metrics. The numerics have shown that the  $G^A$  metrics yields geodesics that resemble the geodesics of the  $G^A$  metric for planar curves from [16, section 5.2]. Namely, when the two spheres are sufficiently far apart, the geodesic passes through a cigar-like middle shape, see figure 10. As predicted by the theory (see section 8.8) geodesics for very high distances tend to have a similar behavior as  $\text{Vol}^k$  metrics, i.e. the geodesic first shrinks the sphere, then moves it, and then blows it up again (cf. section 11.6).

For metrics weighted by higher factors of mean curvature the above equation for the radius has a positive solution. For these metrics geodesics for translations tend to scale the sphere until it has reached the optimal radius and then translate

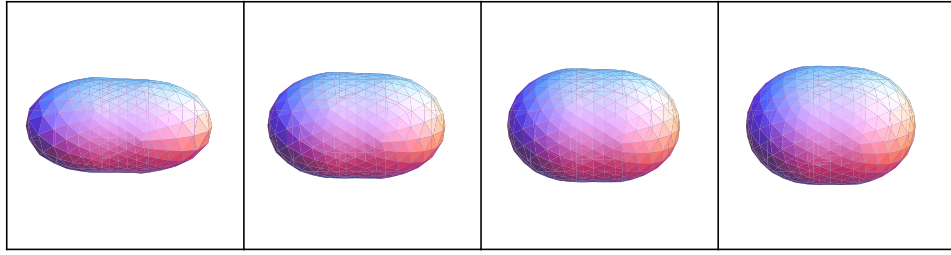


FIGURE 10. Middle figure of a geodesic between two unit spheres translated by distance 3 for  $\Phi = 1 + A \operatorname{Tr}(L)^2$ . From left to right:  $A = 0.2$ ,  $A = 0.4$ ,  $A = 0.6$ ,  $A = 0.8$ . In each of the simulations 20 timesteps and a triangulation with 720 triangles were used.

it. If the radius is already optimal the resulting geodesic is a pure translation (see figure 11).

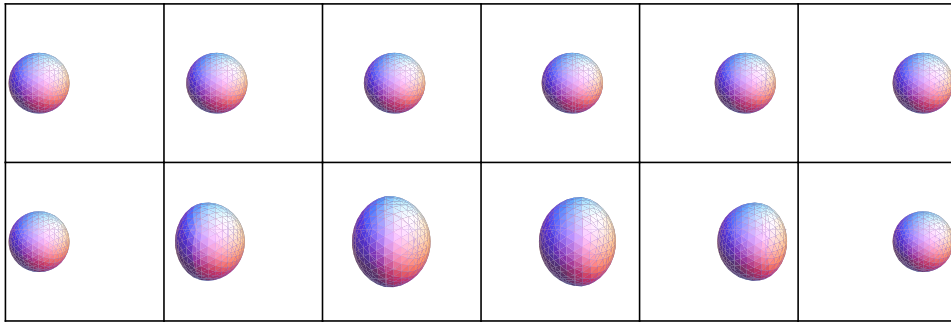


FIGURE 11. Geodesic between two unit spheres translated by distance 3 for  $\Phi = 1 + \frac{1}{16} \operatorname{Tr}(L)^4$  (first row) and  $\Phi = 1 + \operatorname{Tr}(L)^6$  (second row). In each of the experiments 20 timesteps and a triangulation with 720 triangles were used. Time progresses from left to right. Boundary shapes  $t = 0$  and  $t = 1$  are not included.

If the distance is not high enough there still occurs a scaling towards the optimal size, but the middle figure is not a perfect sphere anymore. Instead it is an ellipsoid as in figure 11.

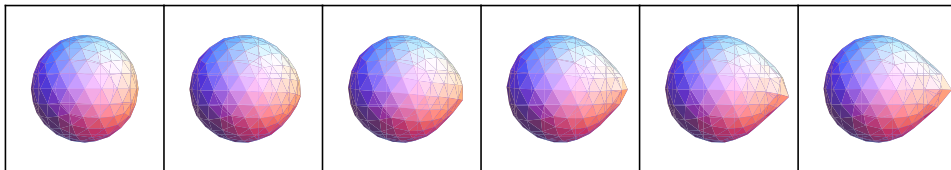


FIGURE 12. Geodesic between a sphere and a sphere with a small bump for  $\Phi = \operatorname{Vol}$ . 20 timesteps and a triangulation with 500 triangles were used. Time progresses from left to right.

**11.8. Deformation of a shape.** We will calculate numerically the geodesic between a shape and a deformation of the shape for various almost local metrics. Small deformations are handled well by all metrics, and they all yield similar results. An example of a geodesic resulting in a small deformation can be seen in figure 12, where a small bump is grown out of a sphere. The energy needed for this deformation is reasonable compared to the energy needed for a pure translation. Taking the metric with  $\Phi = \text{Vol}$  as an example, growing a bump of size 0.4 as in figure 12 costs about a third of a translation of the sphere by 0.4.

Bigger deformations work well with  $\text{Vol}^k$ -metrics and curvature weighted metrics, but not with the  $e^{\text{Vol}}$ -metric, which tends to shrink the object and to concentrate almost all of the deformation at a single time step. In figure 13, a large deformation can be seen for the case of  $\Phi = \text{Vol}$  and  $\Phi = e^{\text{Vol}}$ . Clearly one can see that the  $e^{\text{Vol}}$ -metric concentrates almost all of the deformation in a single time step. We have met this misbehavior of the  $e^{\text{Vol}}$ -metric already with translations. Again, the reason is that  $e^{\text{Vol}}$  is so sensitive to changes in volume.

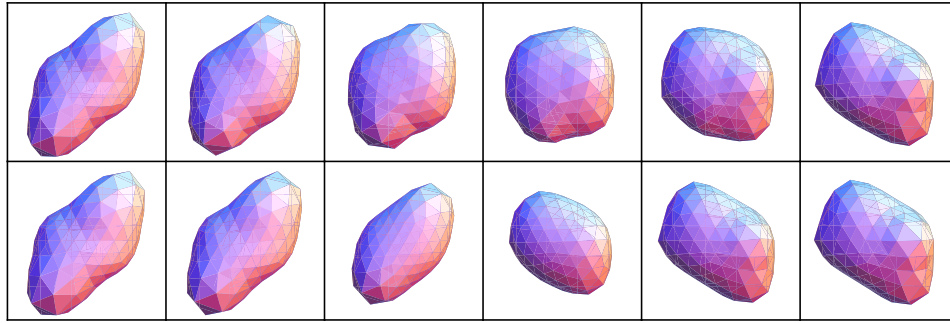


FIGURE 13. Large deformation of a shape for  $\Phi = \text{Vol}$  and  $\Phi = e^{\text{Vol}}$ . 20 timesteps and a triangulation with 500 triangles were used. Time progresses from left to right.

In figure 14 one sees that higher curvature weights smoothen the geodesic.

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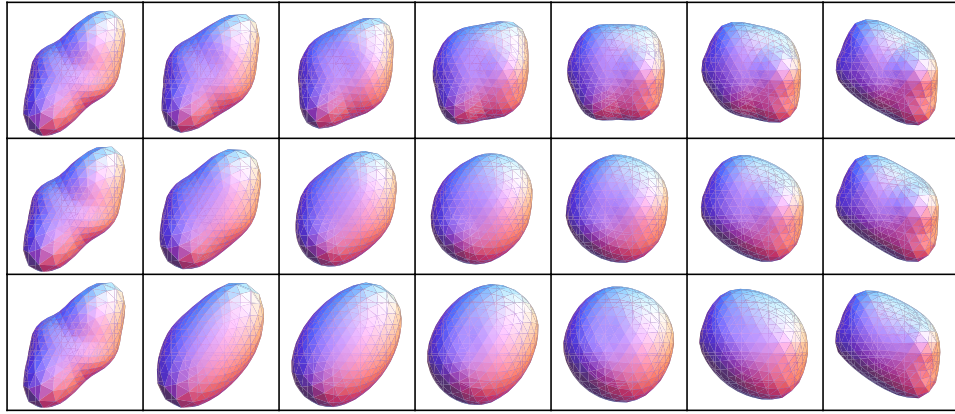


FIGURE 14. Large deformation of a shape for  $\Phi = 1 + 0.1 \operatorname{Tr}(L)^2$  (top),  $\Phi = 1 + 10 \operatorname{Tr}(L)^2$  (middle) and  $\Phi = 1 + \operatorname{Tr}(L)^6$  (bottom). 20 timesteps and a triangulation with 720 triangles were used. Time progresses from left to right.

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# THE AMPL MODEL FILE

LISTING 1. AMPL model file

```

1 param A default 1;
  param k default 1;
3 param B default 1;
  param l default 1;
5 param TimestepsN > 1 integer;
  param VerticesN integer;
7 param PenaltyFactor default 1;
  param PenaltyExponent default 2;
9 set VerticesI := 1..VerticesN;
  set VerticesOfEdgesI within {VerticesI, VerticesI};
11 set VerticesOfFacesI within {VerticesI, VerticesI, VerticesI};
  set FacesOfVerticesI {v in VerticesI} within VerticesOfFacesI;
13 set LinkOfVerticesI {VerticesI} within {VerticesOfFacesI, VerticesOfEdgesI, {-1,1}};
  set AdjacentEdgesOfVerticesI {VerticesI} within {VerticesOfEdgesI, {1,-1}, VerticesOfEdgesI, {1,-1}};
15 set EdgesOfFacesI {VerticesOfFacesI} within VerticesOfEdgesI;
  set EdgesOfVerticesI {v in VerticesI} := setof {(f1,f2,f3,e1,e2,o) in LinkOfVerticesI[v]}(e1,e2);
17
  param Pi default 3.141592653589793;
19 param PerfectAngle {v in VerticesI} default cos(2*Pi/card(FacesOfVerticesI[v]));
  param InitialVertices {VerticesI, 1..3};
21 param FinalVertices {VerticesI, 1..3};

23 var MiddleVertices {2..TimestepsN, VerticesI, 1..3};

25 var Vertices {t in 1..TimestepsN+1, v in VerticesI, i in 1..3} =
  (if t=1 then InitialVertices[v,i]
27   else if t=TimestepsN+1 then FinalVertices[v,i]
```

```

    else MiddleVertices[t,v,i]);
29
var VectorOfEdges {t in 1..TimestepsN+1, (v1,v2) in VerticesOfEdgesI,i in 1..3} =
31   Vertices[t,v2,i] - Vertices[t,v1,i];

var LengthOfEdges {t in 1..TimestepsN+1, (v1,v2) in VerticesOfEdgesI} =
33   sqrt(VectorOfEdges[t,v1,v2,1]^2+VectorOfEdges[t,v1,v2,2]^2+VectorOfEdges[t,v1,v2,3]^2);
35

var CrossOfFaces {t in 1..TimestepsN+1,(v1,v2,v3) in VerticesOfFacesI,i in 1..3} =
37   if i=1 then (Vertices[t,v2,2]-Vertices[t,v1,2])*( Vertices[t,v3,3]-Vertices[t,v1,3]) -
      (Vertices[t,v2,3]-Vertices[t,v1,3])*( Vertices[t,v3,2]-Vertices[t,v1,2])
39   else if i=2 then -(Vertices[t,v2,1]-Vertices[t,v1,1])*( Vertices[t,v3,3]-Vertices[t,v1,3]) +
      (Vertices[t,v2,3]-Vertices[t,v1,3])*( Vertices[t,v3,1]-Vertices[t,v1,1])
41   else (Vertices[t,v2,1]-Vertices[t,v1,1])*( Vertices[t,v3,2]-Vertices[t,v1,2]) -
      (Vertices[t,v2,2]-Vertices[t,v1,2])*( Vertices[t,v3,1]-Vertices[t,v1,1]) ;
43

var NormCrossOfFaces {t in 1..TimestepsN+1,(v1,v2,v3) in VerticesOfFacesI} =
45   sqrt(CrossOfFaces[t,v1,v2,v3,1]^2 + CrossOfFaces[t,v1,v2,v3,2]^2 + CrossOfFaces[t,v1,v2,v3,3]^2);

47 var NuOfFaces {t in 1..TimestepsN+1,(v1,v2,v3) in VerticesOfFacesI,i in 1..3} =
   CrossOfFaces[t,v1,v2,v3,i]/NormCrossOfFaces[t,v1,v2,v3];
49

var AreaOfFaces {t in 1..TimestepsN+1,(v1,v2,v3) in VerticesOfFacesI} =
51   NormCrossOfFaces[t,v1,v2,v3]/2;

53 var AreaOfVertices {t in 1..TimestepsN+1, v in VerticesI} =
   (sum {(f1,f2,f3) in FacesOfVerticesI[v]} AreaOfFaces[t,f1,f2,f3])/3;
55

var VectorAreaOfVertices {t in 1..TimestepsN+1, v in VerticesI, i in 1..3} =
57   (sum {(v1,v2,v3) in FacesOfVerticesI[v]} CrossOfFaces[t,v1,v2,v3,i])/6;

```



```

59 var SquareOfNormOfVectorAreaOfVertices {t in 1..TimestepsN+1, v in VerticesI} =
    VectorAreaOfVertices[t,v,1]^2+VectorAreaOfVertices[t,v,2]^2+VectorAreaOfVertices[t,v,3]^2;
61
var NormOfVectorAreaOfVertices {t in 1..TimestepsN+1, v in VerticesI} =
63     sqrt( SquareOfNormOfVectorAreaOfVertices[t,v]);
65
var Volume {t in 1..TimestepsN+1} =
    sum{(v1,v2,v3) in VerticesOfFacesI} AreaOfFaces[t,v1,v2,v3];
67
var VectorMeanCurvatureOfVertices {t in 1..TimestepsN+1, v in VerticesI, i in 1..3} =
69     if i=1 then
        sum {(f1,f2,f3,e1,e2,o) in LinkOfVerticesI[v]} o*
71         ( VectorOfEdges[t,e1,e2,2]*NuOfFaces[t,f1,f2,f3,3] -
            VectorOfEdges[t,e1,e2,3]*NuOfFaces[t,f1,f2,f3,2] )
73     else if i=2 then
        sum {(f1,f2,f3,e1,e2,o) in LinkOfVerticesI[v]} o*
75         (-VectorOfEdges[t,e1,e2,1]*NuOfFaces[t,f1,f2,f3,3] +
            VectorOfEdges[t,e1,e2,3]*NuOfFaces[t,f1,f2,f3,1] )
77     else
        sum {(f1,f2,f3,e1,e2,o) in LinkOfVerticesI[v]} o*
79         ( VectorOfEdges[t,e1,e2,1]*NuOfFaces[t,f1,f2,f3,2] -
            VectorOfEdges[t,e1,e2,2]*NuOfFaces[t,f1,f2,f3,1] ) ;
81
var SquareOfScalarMeanCurvatureOfVertices {t in 1..TimestepsN+1, v in VerticesI} =
83     (VectorMeanCurvatureOfVertices[t,v,1]^2+VectorMeanCurvatureOfVertices[t,v,2]^2
        +VectorMeanCurvatureOfVertices[t,v,3]^2)/SquareOfNormOfVectorAreaOfVertices[t,v];
85
var PhiOfVertices {t in 1..TimestepsN+1,v in VerticesI} =
87     1+ A*(SquareOfScalarMeanCurvatureOfVertices[t,v])^k +B*(Volume[t])^l;

```

```

89 var IncrementsOfVertices {t in 1..TimestepsN,v in VerticesI,i in 1..3} =
    TimestepsN*(Vertices[t+1,v,i] - Vertices[t,v,i]);
91
var Energy = 1/ 12 / TimestepsN *(
93   sum {t in 1..TimestepsN,v in VerticesI}
      PhiOfVertices[t,v] * sum {(w1,w2,w3) in FacesOfVerticesI[v]}
95     ( IncrementsOfVertices[t,v,1]*CrossOfFaces[t,w1,w2,w3,1] +
        IncrementsOfVertices[t,v,2]*CrossOfFaces[t,w1,w2,w3,2] +
97     IncrementsOfVertices[t,v,3]*CrossOfFaces[t,w1,w2,w3,3] )^2 /
      NormCrossOfFaces[t,w1,w2,w3] +
99   sum {t in 1..TimestepsN,v in VerticesI}
      PhiOfVertices[t+1,v] * sum {(w1,w2,w3) in FacesOfVerticesI[v]}
101     ( IncrementsOfVertices[t,v,1]*CrossOfFaces[t+1,w1,w2,w3,1] +
        IncrementsOfVertices[t,v,2]*CrossOfFaces[t+1,w1,w2,w3,2] +
103     IncrementsOfVertices[t,v,3]*CrossOfFaces[t+1,w1,w2,w3,3] )^2 /
      NormCrossOfFaces[t+1,w1,w2,w3] );
105
var Penalty =
107   sum {t in 1..TimestepsN+1, v in VerticesI,(v1,w1,o1,v2,w2,o2) in AdjacentEdgesOfVerticesI[v]}
      abs(
109     ( VectorOfEdges[t,v1,w1,1]*VectorOfEdges[t,v2,w2,1] +
        VectorOfEdges[t,v1,w1,2]*VectorOfEdges[t,v2,w2,2] +
111     VectorOfEdges[t,v1,w1,3]*VectorOfEdges[t,v2,w2,3] ) * o1 * o2
        / LengthOfEdges[t,v1,w1] / LengthOfEdges[t,v2,w2]
113     - PerfectAngle[v]
        )^PenaltyExponent;
115
minimize f:
117   Energy+Penalty*PenaltyFactor;

```

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